Summary of Aerodynamics A Formulas

1 Relations between height, pressure, density and temperature

1.1 Definitions

g = Gravitational acceleration at a certain altitude ($g_0 = 9.81 m/s^2$) (m/s^2)

r = Earth radius ($6378 km$) (m)

h_g = Height above the ground (Geometric height) (m)

h_a = Height above the center of the earth ($h_a = h_g + r$) (m)

h = Geopotential altitude (Geopotential height) (m)

p = Pressure ($Pa = N/m^2$)

ρ = Air density (kg/m^3)

$\nu = \frac{1}{\rho}$ = Specific volume (m^3/kg)

T = Temperature (K)

$R = 287.05 J/(kgK)$ = Gas constant

$P_s = 1.01325 \times 10^5 N/m^2$ = Pressure at sea level

$\rho_s = 1.225 kg/m^3$ = Air density at sea level

$T_s = 288.15 K$ = Temperature at sea level

$a = \frac{dT}{dh}$ = Temperature gradient ($a = 0.0065 K/m$ in the troposphere (lowest part) of the earth atmosphere) (K/m)

1.2 Relation between geopotential height and geometric height

Newton's gravitational law implicates:

$$g = g_0 \left(\frac{r}{h_a}\right)^2 = g_0 \left(\frac{r}{r + h_g}\right)^2$$

The hydrostatic equation is:

$$dp = -\rho g dh_g$$

However, g is variable here for different heights. Since a variable gravitational acceleration is difficult to work with, the geopotential height h has been introduced such that:

$$dp = -\rho g_0 dh \tag{1.2.1}$$

So this means that:

$$dh = \frac{g}{g_0} dh_g = \frac{r^2}{(r + h_g)^2} dh_g$$

And integration gives the general relationship between geopotential height and geometric height:

$$h = \frac{r}{r + h_g} h_g \tag{1.2.2}$$

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1.3 Relations between pressure, density and height

The famous equation of state is:

$$p = \rho RT \tag{1.3.1}$$

Dividing the hydrostatic equation (1.2.1) by the equation of state (1.3.1) gives as results:

$$\frac{dp}{p} = \frac{-\rho g_o dh}{\rho RT} = -\frac{g_o}{RT} dh$$

If we assume an isothermal environment (the temperature stays the same), then integration gives:

$$\int_{p_1}^{p_2} \frac{dp}{p} = -\frac{g_o}{RT} \int_{h_1}^{h} dh$$

Solving this gives the following equation:

$$\frac{p_2}{p_1} = e^{-\left(\frac{g_0}{RT}\right)(h_2 - h_1)} \tag{1.3.2}$$

And combining this with the equation of state gives the following equation:

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1} \frac{TR}{TR} = \frac{p_2}{p_1} = e^{-\left(\frac{g_0}{RT}\right)(h_2 - h_1)} \tag{1.3.3}$$

1.4 Relations between pressure, density and temperature

We now again divide the hydrostatic equation (1.2.1) by the equation of state (1.3.1), but this time we don't assume an isothermal environment, but we substitute $dh = \frac{dT}{a}$ in it, to get:

$$\frac{dp}{p} = \frac{-\rho g_o dh}{\rho RT} = -\frac{g_o}{aR} \frac{dT}{T}$$

Integration gives:

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{-\frac{g_0}{aR}} \tag{1.4.1}$$

Which is a nice formula. But by using the equation of state, we can also derive the following:

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1} \frac{T_1 R}{T_2 R} = \frac{p_2}{p_1} \left(\frac{T_2}{T_1}\right)^{-1} = \left(\frac{T_2}{T_1}\right)^{-\frac{g_0}{aR}} \left(\frac{T_2}{T_1}\right)^{-1} = \left(\frac{T_2}{T_1}\right)^{-\left(\frac{g_0}{aR}+1\right)} \tag{1.4.2}$$

All those relations can be written in a simpler way:

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\frac{g_0}{g_0 + aR}} = \left(\frac{T_2}{T_1}\right)^{-\frac{g_0}{aR}} \tag{1.4.3}$$

These relations are the standard atmospheric relations in gradient layers.

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2 Continuity equation and Bernoulli's equation

2.1 Definitions

$\dot{m}$ = Mass flow (kg/s)
ρ = Air density (kg/m^3)
A = Area (m^2)
V = Speed (m/s)
p = Pressure $(Pa = N/m^2)$
h = Height (m)
S = Surface (m^2)
r = Radius of the curvature (m)
v = Volume (m^3)
V_a = Velocity difference (m/s)
S_d = Actuator disk surface area (m^2)
T = Thrust (N)
P = Power $(W = J/s)$
η = Efficiency (dimensionless)

2.2 Continuity equation

The continuity equation states that in a tube, the following must be true:

$$\dot{m}_{in} = \dot{m}_{out}$$

This equation can be rewritten to:

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \tag{2.2.1}$$

Or for incompressible flows (where $\rho_1 = \rho_2$):

$$A_1 V_1 = A_2 V_2 \tag{2.2.2}$$

2.3 Bernoulli's equation

The Euler equation, when gravity forces and viscosity are neglected, is:

$$dp = -\rho V dV \tag{2.3.1}$$

This formula is also valid for compressible flows. Integration for 2 points along a streamline gives:

$$(p_2 - p_1) + \rho\left(\frac{1}{2}V_2^2 - \frac{1}{2}V_1^2\right) = 0$$

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This can be easily transformed to the Bernoulli equation, which says that the total pressure is constant along a streamline:

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 \tag{2.3.2}$$

However, if gravity forces are included, the following variant of the Bernoulli equation can be derived:

$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g h_2 \tag{2.3.3}$$

But do remember that the Bernoulli equation is only valid for inviscid (frictionless) incompressible flows.

2.4 Bernoulli's equation of curved flow

When looking at an infinitely small part of air, there is a pressure p working on one side of it, and a pressure $p + dp$ working on the other side. This pressure difference causes the flow to bend. If S is the surface and dr is the length of the part of air, then the resultant force on the part of air is:

$$F = ((p + dp) - p) \cdot S = dp \cdot S = \frac{dp}{dr} \cdot S \cdot dr = \frac{dp}{dr} v$$

Where v is the volume of the air. However, when performing a circular motion, the resultant force is:

$$F = \frac{mV^2}{r} = \frac{\rho V^2}{r} v$$

Combining these data, we find:

$$\frac{dp}{dr} = \frac{\rho V^2}{r} \tag{2.4.1}$$

And therefore the following formula remains constant along a curving flow:

$$\int_{p_1}^{p_2} dp = \int_{r_1}^{r_2} \frac{\rho V^2}{r} dr \tag{2.4.2}$$

And this is the exact reason why concave shapes have a lower pressure in flows, and convex areas have a higher pressure.

2.5 Actuator disk thrust

Suppose we have a propeller, blowing air from left to right. Let's call point 0 a point infinitely far to the left (undisturbed flow), point 1 just to the left of the propeller (infinitely close to it), point 2 identical to point 1, but then on the right, and point 3 identical to point 0, but also on the right. In every point n, the airflow has a pressure p_n, a velocity V_n and an area S_n. Since point 1 and 2 are infinitely close to each other, the air velocity in both points is equal, so $V_1 = V_2$. Since point 0 and point 3 are both in the undisturbed flow, the pressure in those points is equal, so $p_0 = p_3$. Let's define $V_{a_1} = V_1 - V_0$, $V_{a_2} = V_3 - V_2$ and $V_{a_t} = V_3 - V_0$, so $V_{a_t} = V_{a_1} + V_{a_2}$. From the Bernoulli equation, the following equations can be derived:

$$p_0 + \frac{1}{2}\rho V_0^2 = p_1 + \frac{1}{2}\rho (V_0 + V_{a_1})^2$$

$$p_2 + \frac{1}{2}\rho (V_0 + V_{a_1})^2 = p_0 + \frac{1}{2}\rho (V_0 + V_{a_t})^2$$

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Note that it is not allowed to use bernoulli between point 1 and point 2, since energy is added to the flow there. Combining these two equations, we find the pressure difference:

$$p_2 - p_1 = \rho V_{a_t}(V_0 + \frac{1}{2}V_{a_t})$$

If S_d is the surface area of the actuator disk, then the thrust is:

$$T = S_d(p_2 - p_1)$$

Combining this equation with the previous one gives:

$$T = \rho S_d(V_0 + \frac{1}{2}V_{a_t})V_{a_t}$$

However, the thrust is also equal to the mass flow times the change in speed. The mass flow is $\rho S_d(V_0 + V_{a_1})$, and the change in speed is simply equal to V_{a_t}. So:

$$T = \rho S_d(V_0 + V_{a_1})V_{a_t} \tag{2.5.1}$$

Combining this equation with the previous one, we find another important conclusion:

$$V_{a_1} = V_{a_2} = \frac{1}{2}V_{a_t} \tag{2.5.2}$$

2.6 Actuator disk efficiency

Now let's look at the power and the efficiency of the actuator disk. The power input is equal to the change in kinetic energy of the air per unit time:

$$P_{in} = \frac{1}{2}\rho S_d(V_0 + V_{a_1})((V_0 + V_{a_3})^2 - V_0^2) = \rho S_d(V_0 + V_{a_1})^2 V_{a_t}$$

The power that really is outputted, is simply equal to the force times the velocity of the airplane:

$$P_{out} = TV_0 = \rho S_d(V_0 + V_{a_1})V_{a_t}V_0$$

Combing these two equations, we can find the propeller efficiency:

$$\eta = \frac{P_{out}}{P_{in}} = \frac{TV_0}{P_{in}} = \frac{\rho S_d(V_0 + V_{a_1})V_{a_t}V_0}{\rho S_d(V_0 + V_{a_1})^2 V_{a_t}}$$

Simplifying this gives:

$$\eta = \frac{V_0}{V_0 + V_{a_1}} = \frac{1}{1 + \frac{V_{a_1}}{V_0}} \tag{2.6.1}$$

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3 Enthalpy, specific heat and isentropic flows

3.1 Definitions

h = Enthalpy (J/kg)

e = The specific internal energy in a gas (J/kg) (**Note:** This is in some books given the name u)

R = Gas constant (Value is: $287J/(kgK)$) $(J/(kgK))$

T = The temperature of a certain amount of gas (K)

p = Pressure $(Pa = N/m^2)$

v = Specific volume (defined as $\frac{1}{\rho}$) (m^3/kg)

δq = The added heat into a certain amount of gas (J)

c = Specific heat $(J/(kgK))$

c_v = Specific heat at constant volume (For air: $c_v = 717J/(kgK)$) $(J/(kgK))$

c_p = Specific heat at constant pressure (For air: $c_p = 1004J/(kgK)$) $(J/(kgK))$

γ = Ratio of specific heats (Value for normal air is: 1.4) (dimensionless)

ρ = Density (kg/m^3)

3.2 Enthalpy

The enthalpy h is defined as follows:

$$h = e + RT = e + pv \tag{3.2.1}$$

To differentiate that, the chain rule must be applied, since neither p or v are constant. So:

$$dh = de + pdv + vdp \tag{3.2.2}$$

3.3 Specific heat

Note: Almost all of the following formulas apply in general, as long as the gas is a perfect gas.

From basic thermodynamics, it can be derived that:

$$\delta q = de + pdv \tag{3.3.1}$$

When adding an amount of heat to 1 kg of a gas, the temperature of the gas increases. The relation is (per definition):

$$c = \frac{\delta q}{dT} \tag{3.3.2}$$

The addition of heat can be done in multiple ways. If the volume is kept constant (thus $dv = 0$), the following formulas apply:

$$c_v = \frac{\delta q}{dT} \tag{3.3.3}$$

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$$c_v dT = \delta q = de + pdv = de$$

$$e = c_v T \tag{3.3.4}$$

But if the pressure is kept constant (thus $dp = 0$), it is slightly more difficult. For that we will use the enthalpy. The following formulas now apply:

$$c_p = \frac{\delta q}{dT} \tag{3.3.5}$$

$$c_p dT = \delta q = de + pdv = dh - vdp = dh$$

$$h = c_p T \tag{3.3.6}$$

Between the two specific heat constants are interesting relationships. Their difference can be derived:

$$c_p - c_v = \frac{c_p T - c_v T}{T} = \frac{h - e}{T} = \frac{RT}{T} = R$$

And their ratio is (per definition):

$$\gamma = \frac{c_p}{c_v} \tag{3.3.7}$$

3.4 Specific heat ratio in isentropic processes

An isentropic process is both an adiabatic process ($\delta q = 0$) as a reversible process (no friction forces). Suppose there is an isentropic process present. It can in that case be shown that:

$$\frac{dp}{p} = -\frac{c_p}{c_v}\frac{dv}{v} = -\gamma \frac{dv}{v}$$

Integrating that equation and working out the results would give us:

$$\ln \frac{p_2}{p_1} = -\gamma \ln \frac{v_2}{v_1}$$

And since $v = \frac{1}{\rho}$, it can be derived that:

$$\frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma}$$

Using the equation of state ($\frac{p}{\rho} = RT$) it can also be derived that:

$$\frac{p_1}{p_2} = \left(\frac{T_1}{T_2}\right)^{\frac{\gamma}{\gamma-1}}$$

And by combining these two formulas, one can also derive the following formula:

$$\frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma} = \left(\frac{T_1}{T_2}\right)^{\frac{\gamma}{\gamma-1}} \tag{3.4.1}$$

These important relations are called the isentropic flow relations, and are only relevant to compressible flow.

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3.5 Energy equation

We still assume an isentropic process (so $\delta q = 0$). Therefore $\delta q = dh - vdp = 0$, and since the Euler equation says that $dp = -\rho V dV$ we know that $dh + v\rho V dV = 0$. However, $v = \frac{1}{\rho}$ so also $dh + V dV = 0$. Integrating, and working out the results of it, would give:

$$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2 \tag{3.5.1}$$

And since $h = c_p T$, also the following equation is true:

$$c_p T_1 + \frac{1}{2}V_1^2 = c_p T_2 + \frac{1}{2}V_2^2 \tag{3.5.2}$$

This equation is called the energy equation.

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4 Applications of the Mach number

4.1 Definitions

ρ = Air density (kg/m^3)
A = Area (m^2)
a = Speed of sound (m/s)
p = Pressure ($Pa = N/m^2$)
R = Gas constant (Value is: $287 J/(kgK)$) ($J/(kgK)$)
T = The temperature of a certain amount of gas (K)
V = Velocity (m/s)
M = Mach number (dimensionless)
γ = Ratio of specific heats (Value for normal air is: 1.4) (dimensionless)
c_v = Specific heat at constant volume (For air: $c_v = 717 J/(kgK)$) ($J/(kgK)$)
c_p = Specific heat at constant pressure (For air: $c_p = 1004 J/(kgK)$) ($J/(kgK)$)
V_{cal} = Calibrated flight velocity (m/s)

4.2 Speed of sound

The speed of sound is an important thing in aerodynamics. A formula to calculate it can be derived. Therefore the continuity equation is used on a sound wave traveling through a tube with constant area:

$$\rho A a = (\rho + d\rho) A (a + da) \Rightarrow a = -\rho \frac{da}{d\rho}$$

Assuming an isentropic flow, it is allowed to fill in the speed of sound in the Euler equation, which gives $dp = -\rho\, a\, da$. Combining this with the previous equation gives:

$$a = \sqrt{\frac{dp}{d\rho}} \tag{4.2.1}$$

This is not a very easy formula to work with though. But by using the isentropic flow relations, it can be derived that $\frac{dp}{d\rho} = \gamma \frac{p}{\rho}$, and therefore:

$$a = \sqrt{\gamma \frac{p}{\rho}} \tag{4.2.2}$$

Using the equation of state, this can be transformed to:

$$a = \sqrt{\gamma R T} \tag{4.2.3}$$

So the speed of sound only depends on the temperature. But when the speed of sound is known, also the Mach number can be calculated:

$$M = \frac{V}{a} \tag{4.2.4}$$

There are three important regions of Mach numbers:

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1. $M < 1$: Subsonic flow
2. $M = 1$: Sonic flow
3. $M > 1$: Supersonic flow

These regions can be split up into more detailed, but less important regions, as can be seen in table 4.1.

Mach number:	$M < 0.3$	$0.3 < M < 1$	$M = 1$	$M \approx 1$	$1 < M < 5$	$5 < M$
Name:	Low subsonic	High subsonic	Sonic	Transonic	Supersonic	Hypersonic

Table 4.1: Mach regions and corresponding names.

4.3 Isentropic flow relations including Mach number

Let point 0 be the stagnation point, and let point 1 be a point in the undisturbed flow. The stagnation speed V_0 is 0. So p_0 is equal to the total pressure in the flow. The energy equation implies that:

$$c_p T_1 + \frac{1}{2} V_1^2 = c_p T_0 \Rightarrow \frac{T_0}{T_1} = 1 + \frac{V_1^2}{2 c_p T_1}$$

It is known that $R = c_p - c_v$ and $\gamma = \frac{c_p}{c_v}$. From that it can be derived that:

$$c_p = \frac{\gamma R}{\gamma - 1}$$

Combining these two equations results in:

$$\frac{T_0}{T_1} = 1 + \frac{\gamma - 1}{2} \frac{V_1^2}{\gamma R T_1}$$

It is also known that $a_1^2 = \gamma R T_1$ and $M_1 \frac{V_1}{a_1}$. Using that data, the previous formula can be transformed to:

$$\frac{T_0}{T_1} = 1 + \frac{\gamma - 1}{2} M_1^2 \tag{4.3.1}$$

And by using the isentropic flow relations, also the following can be deduced:

$$\frac{p_0}{p_1} = \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\frac{\gamma}{\gamma - 1}} \tag{4.3.2}$$

$$\frac{\rho_0}{\rho_1} = \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\frac{1}{\gamma - 1}} \tag{4.3.3}$$

But do keep in mind that p_0 and ρ_0 are the pressure and density, respectively, at the stagnation point (because $V_0 = 0$).

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4.4 Calculating flight velocity

Calculating flight velocity for low subsonic flows is not difficult, because incompressibility can be assumed. Therefore only the formula for the flight velocity at high subsonic speeds is shown. The equations derived in the previous paragraph can be used to find those formulas. But to do that, the Mach number should first be isolated:

$$M_1^2 = \frac{2}{\gamma - 1}\left[\left(\frac{p_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

And using equation 4.2.4, the following relation is evident:

$$V_1^2 = \frac{2a_1^2}{\gamma - 1}\left[\left(\frac{p_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] \tag{4.4.1}$$

However, it is not easy to measure a pressure ratio $\frac{p_0}{p_1}$. Most measuring devices measure a pressure difference $p_0 - p_1$ instead. Therefore the previous equation can be transformed.

$$V_1^2 = \frac{2a_1^2}{\gamma - 1}\left[\left(\frac{p_0 - p_1}{p_1} + 1\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

This formula still has a pressure value p_1 (which is not a pressure difference), and the speed of sound (which depends on T_1, the static temperature in the air around the airplane). Both are rather difficult to measure accurately. Therefore the calibrated airspeed V_{cal} is introduced, and defined as follows:

$$V_{cal}^2 = \frac{2a_s^2}{\gamma - 1}\left[\left(\frac{p_0 - p_1}{p_s} + 1\right)^{\frac{\gamma-1}{\gamma}} - 1\right] \tag{4.4.2}$$

4.5 Supersonic wind tunnels

From the continuity equation, the following can be derived:

$$\rho A V = c \Rightarrow \ln \rho A V = \ln \rho + \ln A + \ln V = \ln c \Rightarrow \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$$

Where c is any constant mass flow. First filling in Euler's equation $\rho = -\frac{dp}{V\,dV}$ and then filling in equation 4.2.1 results in the following:

$$0 = -\frac{d\rho\; V\,dV}{dp} + \frac{dA}{A} + \frac{dV}{V} = -\frac{V\,dV}{a^2} + \frac{dA}{A} + \frac{dV}{V} = -\frac{M^2\,dV}{V} + \frac{dA}{A} + \frac{dV}{V}$$

A final transformation results in the so-called area-velocity relation:

$$\frac{dA}{A} = \left(M^2 - 1\right)\frac{dV}{V} \tag{4.5.1}$$

This formula has interesting implications. When looking at the three previously discussed Mach number regions, the following three phenomena can occur:

1. Subsonic flows ($M < 1$): For the velocity to increase, the area must decrease (and vice-versa).
2. Sonic flows ($M = 1$): This is a rather strange case, because it implies $\frac{dA}{A}$ to be 0. Therefore this only occurs in the tunnel where the cross-sectional area is minimal (this point is called the throat of the tunnel).
3. Supersonic flows ($M > 1$): For the velocity to increase, the area must increase (and vice-versa).

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5 Flow types caused by viscous effects

5.1 Definitions

δ = Boundary layer thickness (m)

τ_w = Shear stress at the surface $(Pa = N/m^2)$

V = Air flow velocity (m/s)

μ = Absolute viscosity coefficient (or short: viscosity) (For normal air ($T = 288K$): $1.7894 \times 10^{-5} kg/(ms)$)

Re_x = The Reynolds number at a position x (dimensionless)

ρ = Air density (kg/m^3)

x = The distance from the leading edge of the air foil (m)

c_{f_x} = Skin friction coefficient (dimensionless)

q = Dynamic pressure $(Pa = N/m^2)$

D_f = Total skin friction drag (N)

b = Wing span (m)

S = Wing surface (m^2)

C_f = Total skin friction drag coefficient (dimensionless)

Re_{cr} = The critical Reynolds number - Reynolds number where transition occurs (dimensionless)

x_{cr} = The critical value - Distance between the leading edge and the transition point (m)

5.2 Boundary layers and surface shear stress

Involving viscous effects, the airflow doesn't move smoothly over the air foil. Instead, a boundary layer appears with thickness δ. In this boundary layer, at a distance of y from the air foil, the flow velocity is V_y. V_0 is 0, and V_δ is equal to the flow speed calculated for frictionless flow, assuming the wing has the shape of the airfoil plus its boundary layer.

A very important number in aerodynamics is the Reynolds number. The Reynolds number at a distance x from the leading edge of the airfoil can be calculated using this formula:

$$Re_x = \frac{V_\infty \rho_\infty x}{\mu_\infty} \tag{5.2.1}$$

Important for drag calculations, is the shear stress caused by the flow. The shear stress at the surface can be calculated using the slope of the V_y-curve at $y = 0$:

$$\tau_w = \mu \left(\frac{dV}{dy} \right)_{y=0} \tag{5.2.2}$$

For laminar flows, the ratio $\frac{dV}{dy}$ is smaller than for turbulent flows, so laminar flows also have a lower surface shear stress compared to turbulent flows. But rather than dealing with shear stress, aerodynamicists find it often easier to work with the dimensionless skin friction coefficient, which is per definition equal to the following:

$$c_{f_x} = \frac{\tau_w}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{\tau_w}{q_\infty} \tag{5.2.3}$$

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For laminar flows, experiments have shown that the following two formulas apply:

$$\delta = \frac{5.2x}{\sqrt{Re_x}} \tag{5.2.4}$$

$$c_{f_x} = \frac{0.664}{\sqrt{Re_x}} \tag{5.2.5}$$

And for turbulent flows, the following applies:

$$\delta = \frac{0.37x}{Re_x^{0.2}} \tag{5.2.6}$$

$$c_{f_x} = \frac{0.0592}{Re_x^{0.2}} \tag{5.2.7}$$

5.3 Skin friction drag

Note: The following formulas only apply for airfoils with the shape of a flat plate. It also only applies for incompressible flows. Otherwise a correction factor, depending on the Mach number, is required.

Using the surface shear stress, the total skin friction drag can be calculated. To find this, the surface shear stress should be integrated over the entire wing area. In formula this is:

$$D_f = \int_0^b \left(\int_0^L c_{f_x} q_\infty dx \right) dw$$

Solving this for laminar flow gives:

$$D_f = \frac{1.328 q_\infty S}{\sqrt{\frac{\rho_\infty V_\infty L}{\mu}}}$$

Note that the distances are now based on the total length of the airfoil. To simplify this equation, the total skin friction drag coefficient C_f (not to be mistaken with the local skin friction coefficient c_{f_x}) is introduced:

$$C_f = \frac{D_f}{q_\infty S} \tag{5.3.1}$$

Using this definition with the total skin friction drag gives:

$$C_f = \frac{1.328}{\sqrt{Re_L}} \tag{5.3.2}$$

And doing all the steps analog for turbulent flow results in:

$$C_f = \frac{0.074}{Re_L^{0.2}} \tag{5.3.3}$$

Research has shown that the transition from laminar to turbulent flow occurs, when the Reynolds number rises above a critical value: the critical Reynolds number Re_{cr}. The point at which that occurs, is the transition point, and the distance from that point to the leading edge is the critical value x_{cr}. The following relation then applies:

$$Re_{cr} = \frac{\rho_\infty V_\infty x_{cr}}{\mu_\infty} \tag{5.3.4}$$

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6 Coefficients and supersonic flows

6.1 Definitions

R = Resultant force that acts on the wing (N)
N = Component of R acting perpendicular to the chord line (N)
A = Component of R acting tangential to the chord line (N)
L = Lift - component of R acting perpendicular to the relative wind (N)
D = Drag - component of R acting parallel to the relative wind (N)
α = Angle of attack (usually the angle between the chord line and the relative wind) (deg)
M = Mach number (dimensionless)
Re = Reynolds number (dimensionless)
c = Chord - the length of the chord line (line between the leading edge and the trailing edge) (m)
c_l = Lift coefficient for infinite wings (dimensionless)
ρ = Air density (kg/m^3)
V = Air velocity (m/s)
S = Wing area (m^2)
q = Dynamic air pressure, defined as $q = \frac{1}{2}\rho V^2$ ($Pa = N/m^2$)
c_d = Drag coefficient for infinite wings (dimensionless)
c_m = Moment coefficient for infinite wings (dimensionless)
C_p = Pressure coefficient (dimensionless)
$C_{p,l}$ = Pressure coefficient on the lower side of the wing (dimensionless)
$C_{p,u}$ = Pressure coefficient on the upper side of the wing (dimensionless)
c_n = Normal force coefficient for infinite wings (dimensionless)
c_a = Axial force coefficient for infinite wings (dimensionless)
M_{cr} = Critical Mach number - Mach number at which shock waves start occurring (dimensionless)
μ = Mach angle - Angle between direction of flight and shock wave boundary (rad/deg)

6.2 Airfoil nomenclature

During flight, a resultant force R acts on the wing. This force can be resolved into two forces in multiple ways. R can be resolved in N and A, where N is the component perpendicular (normal) to the chord, and A is the component tangential (axial) to the chord. However, R can also be resolved in a component L (lift) perpendicular to the relative wind, and a component D (drag) parallel to the relative wind. These forces have the following relation:

$$L = N\cos\alpha - A\sin\alpha \tag{6.2.1}$$

$$D = N\sin\alpha + A\cos\alpha \tag{6.2.2}$$

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6.3 Applying dimensional analysis to infinite wings

Using dimensional analysis it can be found that:

$$L = Z\rho_\infty V_\infty^2 S \left(\frac{1}{M_\infty}\right)^e \left(\frac{1}{Re_c}\right)^f$$

Where Z is a constant (as long as α remains constant, because if α changes, also Z changes). So if we define the lift coefficient c_l such that:

$$\frac{c_l}{2} = Z \left(\frac{1}{M_\infty}\right)^e \left(\frac{1}{Re_c}\right)^f \tag{6.3.1}$$

Then we see that:

$$L = \frac{1}{2}\rho_\infty V_\infty^2 S c_l = q_\infty S c_l \tag{6.3.2}$$

And now we see that the lift coefficient is also equal to:

$$c_l = \frac{L}{q_\infty S} \tag{6.3.3}$$

Doing the same steps for the drag and the moment, will give:

$$D = q_\infty S c_d \tag{6.3.4}$$

$$M = q_\infty S c c_m \tag{6.3.5}$$

And finally we summarize all the equations:

$$c_l = \frac{L}{q_\infty S} \qquad c_d = \frac{D}{q_\infty S} \qquad c_m = \frac{M}{q_\infty S c} \tag{6.3.6}$$

$$c_l = f_1(\alpha, M_\infty, Re) \qquad c_d = f_2(\alpha, M_\infty, Re) \qquad c_m = f_3(\alpha, M_\infty, Re) \tag{6.3.7}$$

Where f_1, f_2 and f_3 are functions. This is to emphasize that the coefficients depend on the parameters noted in brackets.

6.4 Bending coefficients

As was already mentioned in the previous paragraph, the moment coefficient is defined as:

$$c_{m_x} = \frac{M_x}{q_\infty S c} \tag{6.4.1}$$

Where x can be any distance from the leading edge of the wing (so $0 \leq x \leq c$) and M is the moment acting on that point. Note that there is now an extra variable c in the equation, where there was none in the definition for the force coefficients. If the chord c wouldn't be present, c_m wouldn't be dimensionless.

There are two specific points concerning moments which are often used in aerodynamics. The first one is the center of pressure (dp in short). This is where there is no bending moment. The position of this point usually changes if the angle of attack changes. The second point is the aerodynamic center (ac). This is the point where the bending moment stays constant as the angle of attack changes. Its position, which is almost always around the quarter chord position ($\frac{x_{ac}}{c} \approx 0.25$), doesn't change if the angle of attack changes. So in formula:

$$c_{m_{dp}} = 0 \qquad \frac{dc_{m_{ac}}}{d\alpha} = 0 \tag{6.4.2}$$

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So the moment coefficient is different on different positions on an airfoil. But there exists a relationship between those moment coefficients. And, using simple statics, it can be shown that:

$$c_{m_{Q_1}} - c_{m_{Q_2}} = c_n \left(\frac{x_{Q_1}}{c} - \frac{x_{Q_2}}{c} \right) \tag{6.4.3}$$

Do remember that this formula only applies for constant angle of attack, since otherwise variables like c_n change. However, the fact that $\frac{x_{ac}}{c}$ remains constant for different angles of attack can be used.

6.5 Prandtl-Glauert Rule

Let's define a new dimensionless coefficient to indicate the pressure over a wing. We define the pressure coefficient C_p as follows:

$$C_p = \frac{p - p_\infty}{q_\infty} = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} \tag{6.5.1}$$

The pressure coefficient can be plotted for a certain airfoil. Suppose the pressure coefficient at a point on an airfoil at low speeds ($M_\infty \approx 0$) is measured. If the air velocity increases, also the absolute value of the pressure coefficient increases (negative C_p get even more negative). This is, according to the Prandtl-Glauert rule, approximately equal to:

$$C_p = \frac{C_{p,0}}{\sqrt{1 - M_\infty^2}} \tag{6.5.2}$$

Where $C_{p,0}$ is the pressure coefficient at low speeds. For $M_\infty < 0.3$ compressibility effects don't need to be taken into account, and for $M_\infty > 0.7$ this formula loses its accuracy. Therefore this formula is only really applicable for $0.3 < M_\infty < 0.7$.

Now let's define the normal force coefficient and the axial force coefficient for unit length (for just $1m$ of the wing) in the same way as the lift coefficient:

$$c_n = \frac{N}{q_\infty c} \qquad c_a = \frac{A}{q_\infty c} \tag{6.5.3}$$

It can then be shown that the normal force coefficient is also equal to the following integral:

$$c_n = \int_0^1 (C_{p,l} - C_{p,u}) d\frac{x}{c} \tag{6.5.4}$$

Combining this with equation 6.2.1 results in:

$$c_l = c_n \cos\alpha - c_a \sin\alpha \tag{6.5.5}$$

Most aircrafts have their cruising angle of attack at $\alpha < 5°$. And for such small angles of attack, $\sin\alpha \rightarrow 0$ and $\cos\alpha \rightarrow 1$. So $c_l \approx c_n$, and then equation 6.5.4 can also be used to calculate the lift coefficient.

But for this lift coefficient, the Prandtl-Glauert rule can also be applied. If $c_{l,0}$ is the lift coefficient for low air velocities ($M_\infty < 0.3$), then the lift coefficient at higher Mach numbers is:

$$c_l = \frac{c_{l,0}}{\sqrt{1 - M_\infty^2}} \tag{6.5.6}$$

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6.6 Critical pressure coefficient

Using the definition of the pressure coefficient, the definition of the dynamic pressure, the formula for the speed of sound, the isentropic flow relations and a few assumptions, the following formula can be derived:

$$C_p = \frac{2}{\gamma M_\infty^2}\left(\left[\frac{1+\frac{1}{2}(\gamma-1)M_\infty^2}{1+\frac{1}{2}(\gamma-1)M^2}\right]^{\frac{\gamma}{\gamma-1}} - 1\right) \tag{6.6.1}$$

So for a certain atmosphere and a constant free-stream Mach number, the pressure coefficient only depends on the local Mach number M on the wing. To find the critical pressure coefficient, we should fill in $M = 1$. This results in:

$$C_p = \frac{2}{\gamma M_\infty^2}\left(\left[\frac{2+(\gamma-1)M_\infty^2}{\gamma+1}\right]^{\frac{\gamma}{\gamma-1}} - 1\right) \tag{6.6.2}$$

So the critical pressure coefficient only depends on the Mach number, which is quite an interesting thing. To find the critical pressure coefficient, the Prandtl-Glauert rule should be used. If M_{cr} is the critical Mach number, and $C_{p,0}$ is the lowest pressure coefficient on the wing for low air velocities, then the following formula applies:

$$\frac{C_{p,0}}{\sqrt{1-M_{cr}^2}} = C_p = \frac{2}{\gamma M_{cr}^2}\left(\left[\frac{2+(\gamma-1)M_{cr}^2}{\gamma+1}\right]^{\frac{\gamma}{\gamma-1}} - 1\right) \tag{6.6.3}$$

6.7 Supersonic flight

To reduce the critical Mach number M_{cr}, a plane can be equipped with swept wings. Because these wings have a certain angle with respect to the airflow, the speed of the air relative to the leading edge of the wing is reduced. If Ω is the angle between the leading edge of the wing, and the line perpendicular to the direction of flight (so for normal planes $\Omega = 0$), then the new critical Mach number is $\frac{M_{cr}}{\cos\Omega}$. However, to use the component of the air velocity normal to the leading edge of the swept wings is not always accurate. However, the following formula is correct:

$$M_{cr,normal} < M_{cr,swept} < \frac{M_{cr,normal}}{\cos\Omega} \tag{6.7.1}$$

But if supersonic flight does happen, shock waves occur. These shock waves have a certain angle with respect to the direction of flight. For flat plates, this angle can easily be calculated:

$$\mu = \arcsin\frac{1}{M} \tag{6.7.2}$$

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7 Drag and 3D wings

7.1 Definitions

$D_{profile}$ = Profile drag (N)
$D_{friction}$ = Friction drag(N)
$D_{pressure}$ = Pressure (N)
$c_{d,profile}$ = Profile drag coefficient for unit length (dimensionless)
$c_{d,f}$ = Friction drag coefficient for unit length (dimensionless)
$c_{d,p}$ = Pressure drag coefficient for unit length (dimensionless)
D_{wave} = Wave drag(N)
D = Total drag (N)
$c_{d,w}$ = Wave drag coefficient for unit length (dimensionless)
c_d = Drag coefficient for unit length (dimensionless)
α = Angle of attack (deg)
α_{eff} = Effective angle of attack (deg)
α_i = Induced angle of attack (rad)
D_i = Induced drag (N)
L = Lift (N)
C_L = Lift coefficient (dimensionless)
A = Aspect ratio (dimensionless)
b = Wing span (m)
c = Wing chord length (m)
S = Wing area (m^2)
q_∞ = Dynamic pressure in free-stream ($Pa = N/m^2$)
e = Span effectiveness ratio (sometimes also called Oswald factor) (dimensionless)

7.2 Drag types for 2D airfoils

There are three important types of drag in aerodynamics. Skin friction drag has already been discussed in a previous chapter, and so does pressure drag due to flow separation. Together these two types of drag form the profile drag. In formula:

$$D_{profile} = D_{friction} + D_{pressure} \tag{7.2.1}$$

$$c_{d,profile} = c_{d,f} + c_{d,p} \tag{7.2.2}$$

But there is another type of drag, called wave drag. This is caused by shock waves, which are caused by supersonic velocities. So the total drag is:

$$D = D_{wave} + D_{profile} = D_{wave} + D_{friction} + D_{pressure} \tag{7.2.3}$$

$$c_d = c_{d,w} + c_{d,f} + c_{d,p} \tag{7.2.4}$$

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7.3 Induced Drag

Induced drag doesn't occur in 2-dimensional airfoils. In 3-dimensional airfoils it does appear. And since airplanes have 3-dimensional airfoils, it plays an important role. It usually occurs that the local flow direction of the air differs from the relative wind. Therefore the effective angle of attack α_{eff} is smaller than the geometric angle of attack α. Their difference is α_i, the induced angle of attack. In formula:

$$\alpha_i = \frac{\pi}{180}(\alpha - \alpha_{eff}) \tag{7.3.1}$$

Note that a conversion factor is necessary. This is because α_i is in radians (this is necessary for equation 7.3.3), while the normal angle of attack is in degrees.

Geometrically it can be shown that:

$$D_i = L \sin \alpha_i = L\alpha_i \tag{7.3.2}$$

The latter part of the equation is an approximation, since α_i is very small, and therefore $\sin \alpha_i \approx \alpha_i$. For elliptical lift distribution, which is often approximately the case for airplanes, the following formula is true for incompressible flows:

$$\alpha_i = \frac{C_L}{\pi A} \tag{7.3.3}$$

Where the aspect ratio A is equal to the 'slenderness' of the wing $\frac{b}{c}$. However, c is not constant along the wing, so aerodynamicists therefore have defined the aspect ratio as:

$$A = \frac{b^2}{S} \tag{7.3.4}$$

Combining previous equations results in:

$$D_i = L\frac{C_L}{\pi A} = q_\infty S \frac{C_L^2}{\pi A} \tag{7.3.5}$$

So now the induced drag coefficient can be found:

$$C_{D,i} = \frac{C_L^2}{\pi A} \tag{7.3.6}$$

However, elliptical lift distributions aren't always the case. Therefore, the span efficiency factor e (also sometimes called Oswald factor) has been defined, such that:

$$C_{D,i} = \frac{C_L^2}{\pi A e} \tag{7.3.7}$$

Now let's calculate the total drag coefficient for the wing. We don't know the induced drag for supersonic speeds, so for (low) subsonic speeds, the following equation holds:

$$C_D = c_{d,profile} + \frac{C_L^2}{\pi e A} = c_{d,f} + c_{d,p} + \frac{C_L^2}{\pi e A} \tag{7.3.8}$$

7.4 Slope of the (C_L - α) curve

For 2D wings, the slope of the (C_L - α) curve is $\frac{dC_L}{d\alpha} = a_0$. This is not the case for 3D wings. Now the relation $\frac{dC_L}{\alpha_{eff}} = a_0$ holds. Using equations 7.3.1 and 7.3.3 (the latter with a new span effectiveness factor e_1), and solving it for C_L gives the following relation:

$$a = \frac{a_0}{1 + \frac{57.3 a_0}{\pi e_1 A}} \tag{7.4.1}$$

e_1 is in theory a different factor than e, but in practice they are approximately equal. The factor 57.3 is in fact $\frac{180}{\pi}$, a conversion factor that was used to convert α_i to radians in the derivation of this formula.

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8 Dimensional Analysis

8.1 Definitions

L = Lift (N)
V = Air velocity (m/s)
ρ = Density (kg/m^3)
S = Wing surface (m^2)
a = Speed of sound (m/s)
μ = Viscosity ($kg/(m\,s)$)

8.2 Introduction to Dimensional Analysis

Suppose the value of a certain variable x_1 depends only on the variables $x_2, x_3, \ldots, x_k$. So $x_1 = f(x_2, x_3, \ldots, x_k)$. Now assume that the function f is a function, such that $x_1 = \phi\, x_2^{p_2}\, x_3^{p_3}\, \ldots\, x_k^{p_k}$, where ϕ and p_i are constant coefficients for every i, then the units of both sides must be equal. This is the assumption dimensional analysis is based on.

Suppose k is the amount of variables you have. These variables all have certain units. However, there is often a relation between units. For example, $1N = 1\frac{kg\,m}{s^2}$ according to Newton's second law. All the variables have to have their units set back to certain more or less basic units. It is normal to choose M, L and T for that, for mass, length and time respectively. Suppose r is the amount of basic units needed. Usually $r = 3$, but sometimes $r = 2$ or $r = 4$. In rare cases r can even get up to 7, but this is rather unlikely to occur. In table 8.1 certain variables have been expressed in their basic components.

Units	F (N)	p (Pa)	μ ($\frac{kg}{m\,s}$)	ρ ($\frac{kg}{m^3}$)	V ($\frac{m}{s}$)
Basic Units	MLT^{-2}	$ML^{-1}T^{-2}$	$ML^{-1}T^{-1}$	ML^{-3}	LT^{-1}

Table 8.1: Variables in their "basic" components.

8.3 Buckingham Pi-Theorem generally explained

To use the Buckingham Pi-Theorem, the following steps should be taken. These steps might seem vague, but comparing the general steps with the example of the next paragraph might offer some clarity at times.

- You first need to write down all the depending variables $x_1, x_2, \ldots, x_k$ and their corresponding units resolved into the basic units.
- Now pick r repeating variables (for the definition of r, see previous paragraph). It is often wise to pick variables that often appear in well-known dimensionless coefficients. The velocity V is a good example, and also ρ and S are often used in the case of aerodynamics. Dimensional analysis with other variables will work just fine as well, but the outcome may be slightly different. You should not pick x_1 (the variable for which you want to find a function) though. For simplicity we will assume that $r = 3$ (if it's not, the steps are still the same). So let's suppose you've picked x_2, x_3 and x_4 as repeating variables.

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- Now do the following steps $k - r$ (so for us this would be $k - 3$) times:
  - Suppose this is the i'th time we've performed these steps. Pick one of the non-repeating variables (one you haven't chosen yet). If $i = 1$ (so if this is the first time you're performing these steps) you should pick x_1. Let's assume you've picked x_j. Now write down the formula:

$$\Pi_i = x_j x_2^b x_3^c x_4^d \tag{8.3.1}$$

    b, c and d are just dimensionless coefficients. We have not taken a in this case, because in the example of the next paragraph a is already used for the speed of sound, and 2 different meanings of a is rather confusing.
  - Then assume that the dimensions of both side of the equation are equal. Π_i is dimensionless. Suppose x_n for every n has as units $M^{m_n} L^{l_n} T^{t_n}$, then write down the following equation (which must be true):

$$M^0 L^0 T^0 = M^{m_j} L^{l_j} T^{t_j} \left(M^{m_2} L^{l_2} T^{t_2}\right)^b \left(M^{m_3} L^{l_3} T^{t_3}\right)^c \left(M^{z_4} L^{z_4} T^{z_4}\right)^d \tag{8.3.2}$$

  - Now write down the following r equations (in our case 3):

$$\begin{aligned} 0 &= m_j + bm_2 + cm_3 + dm_4 \\ 0 &= l_j + bl_2 + cl_3 + dl_4 \\ 0 &= t_j + bt_2 + ct_3 + dt_4 \end{aligned} \tag{8.3.3}$$

  - Solve these r equations for a, b and c (if $r = 4$ you should solve it for a, b, c and d of course). Then write down the following:

$$\Pi_i = x_j x_2^b x_3^c x_4^d \tag{8.3.4}$$

    But instead of writing down b, c and d you write down the values of it. Now you've found one of the Π-terms. To find the other Π-terms just repeat the previous steps.

- When you've found all of the $k - r$ Π-terms, just write down the following:

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots, \Pi_{k-r}) \tag{8.3.5}$$

  But instead of writing the Π's, fill in the "values" you've found in the previous steps. Then the dimensional analysis is completed. Of course it is wise to check whether all the Π-terms are dimensionless.

8.4 Applying the Buckingham Pi-Theorem

To show how to apply the Buckingham Pi-Theorem, we use an example. Let's use Dimensional Analysis to derive a formula for the lift L, depending on the variables V, ρ, S, a and μ. These are all our variables. There are 6 variables, so $k = 6$. In table 8.2 the basic units of these variables have been written down.

Variable	L	V	ρ	S	a	μ
Basic Units	MLT^{-2}	LT^{-1}	ML^{-3}	L^2	LT^{-1}	$ML^{-1}T^{-1}$

Table 8.2: Writing down the variables for the example expressed in basic units.

We now need to choose repeating variables. Note that we have 3 basic units (and we can't simplify it any further), so $r = 3$. We now also know that we need $k - r = 6 - 3 = 3$ Π-terms. Let's choose V, ρ and S as the repeating variables. Now we should find the Π-terms:

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1. The first Π-term. Since we want to find a function for L, and since this is the first Π-term, we should pick L as non-repeating variable. Now we should write down the following:

$$\Pi_1 = LV^b \rho^c S^d \tag{8.4.1}$$

Now we write this in the basic units:

$$M^0 L^0 T^0 = MLT^{-2} \left(LT^{-1}\right)^b \left(ML^{-3}\right)^c \left(L^2\right)^d \tag{8.4.2}$$

And we write the equations we need to solve, for M, L and T respectively:

$$\begin{aligned} 0 &= 1 + 0 + c + 0 \\ 0 &= 1 + b - 3c + 2d \\ 0 &= -2 - b + 0 + 0 \end{aligned} \tag{8.4.3}$$

Solving this gives $b = -2$, $c = -1$ and $d = -1$. Now we have found our first Π-term, so we should write it down according to equation 8.4.1:

$$\Pi_1 = LV^{-2}\rho^{-1}S^{-1} = \frac{L}{V^2 \rho S} \tag{8.4.4}$$

2. The second Π-term. Now we pick a as non-repeating variable, and perform the steps identically:

$$\Pi_2 = aV^b \rho^c S^d \tag{8.4.5}$$

$$M^0 L^0 T^0 = LT^{-1} \left(LT^{-1}\right)^b \left(ML^{-3}\right)^c \left(L^2\right)^d \tag{8.4.6}$$

$$\begin{aligned} 0 &= 0 + 0 + c + 0 \\ 0 &= 1 + b - 3c + 2d \\ 0 &= -1 - b + 0 + 0 \end{aligned} \tag{8.4.7}$$

Solving gives $b = -1$, $c = 0$ and $d = 0$. So:

$$\Pi_2 = \frac{a}{V} \tag{8.4.8}$$

3. The third Π-term. Only one non-repeating variables remains. So we pick μ, and do the same steps once more.

$$\Pi_3 = \mu V^b \rho^c S^d \tag{8.4.9}$$

$$ML^{-1}T^{-1} = LT^{-1} \left(LT^{-1}\right)^b \left(ML^{-3}\right)^c \left(L^2\right)^d \tag{8.4.10}$$

$$\begin{aligned} 0 &= 1 + 0 + c + 0 \\ 0 &= -1 + b - 3c + 2d \\ 0 &= -1 - b + 0 + 0 \end{aligned} \tag{8.4.11}$$

Solving gives $b = -1$, $c = -1$ and $d = -\frac{1}{2}$. So:

$$\Pi_3 = \frac{\mu}{V\rho\sqrt{S}} \tag{8.4.12}$$

All the Π-terms have been written down, so we can write down the function:

$$\Pi_1 = f(\Pi_2, \Pi_3) \tag{8.4.13}$$

Filling in the Π-terms gives:

$$\frac{L}{V^2 \rho S} = f\left(\frac{a}{V}, \frac{\mu}{V\rho\sqrt{S}}\right) \tag{8.4.14}$$

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This is where the Buckingham Pi-Theorem usually ends. However, in this case we can derive a bit more data. Note that Π_2 is 1 over the Mach number, and Π_3 is 1 over the Reynolds number (since $\sqrt{S}$ is a length). So also the following is true:

$$\frac{L}{V^2 \rho S} = f(M, Re) = \phi M^e Re^f \tag{8.4.15}$$

Where ϕ, e and f are dimensionless constants. If we define $\frac{C_L}{2} = \phi M^e Re^f$, then:

$$L = C_L \frac{1}{2} \rho V^2 S \tag{8.4.16}$$

And this is a formula that should look familiar to you.

8.5 Different method of using Dimensional Analysis

Next to the Buckingham Pi-Theorem there is another method of using dimensional analysis, which is rather similar. This method is often faster, but there are a few cases where you should be very careful not to make assumptions you're not allowed to make. So even though the Buckingham Pi-Theorem is often a bit more work, it is a safer method. And if you feel you're having trouble understanding the Buckingham Pi-Theorem, the following 2 paragraphs might just confuse you and may not be worth reading.

Using this method goes as follows. First write down the following equation:

$$x_1 = \phi \, x_2^{p_2} \, x_3^{p_3} \, \ldots \, x_k^{p_k} \tag{8.5.1}$$

Where ϕ and p_i (for every i) are just dimensionless coefficients. Now write this down in units, more or less identical as in the paragraph introducing the Buckingham Pi-Theorem. For simplicity we will once more assume that $r = 3$, and that the basic units are M, L and T.

$$M^{m_1} L^{l_1} T^{t_1} = \phi \left(M^{m_2} L^{l_2} T^{t_2}\right)^{p_2} \left(M^{m_3} L^{l_3} T^{t_3}\right)^{p_3} \ldots \left(M^{m_k} L^{l_k} T^{t_k}\right)^{p_k} \tag{8.5.2}$$

Now write the r corresponding coefficient equations:

$$\begin{aligned} m_1 &= p_2 m_2 + p_3 m_3 + \ldots + p_k m_k \\ l_1 &= p_2 t_2 + p_3 l_3 + \ldots + p_k l_k \\ t_1 &= p_2 l_2 + p_3 t_3 + \ldots + p_k t_k \end{aligned} \tag{8.5.3}$$

Note that the only unknowns in these equations are the p-coefficients. Normally we would choose x_2, x_3 and x_4 as repeating variables. However, there are no repeating variables in this method, but instead of choosing x_2, x_3 and x_4 as repeating variables, we just solve the previous 3 equations for p_2, p_3 and p_4, thus expressing them in the other p-coefficients. We then fill those values in, in equation 8.5.1. Then we simplify the equation we get, by bonding all the variables to the power p_i together in brackets for every i (except for the i you have chosen as "repeating"). This probably seems vague, but it might become clearer when you read the example of the next paragraph. But when this "bonding" is done, the equation found should be the same as gotten with the Buckingham-Pi Theorem, as long as the same "repeating" variables were chosen.

8.6 Example of the second method

We will now demonstrate the second method of using dimensional analysis. We will do this by once more deriving a formula for the lift L. According to the method described in the previous paragraph, and according to table 8.2, we can write down the following equation:

$$L = \phi V^b \rho^c S^d a^e \mu^f \tag{8.6.1}$$

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In units this is:

$$MLT^{-2} = \left(LT^{-1}\right)^b \left(ML^{-3}\right)^c \left(L^2\right)^d \left(LT^{-1}\right)^e \left(ML^{-1}T^{-1}\right)^f \tag{8.6.2}$$

The three equations we get are:

$$\begin{aligned} 1 &= 0 + c + 0 + 0 + f \\ 1 &= b - 3c + 2d + e - f \\ -2 &= -b - e - f \end{aligned} \tag{8.6.3}$$

Normally we would choose V, ρ and S as repeating variables. Therefore we now solve these equations for b, c and d. This results in $b = 2 - e - f$, $c = 1 - f$ and $d = 1 - \frac{f}{2}$. So now we can write down an equation. And according to the described method we should bond variables with the same power in brackets as follows:

$$L = \phi V^{2-e-f} \rho^{1-f} S^{1-\frac{f}{2}} a^e \mu^f = \phi \frac{V^2}{V^e V^f} \frac{\rho}{\rho^f} \frac{S}{S^{\frac{f}{2}}} a^e \mu^f = \phi V^2 \rho S \left(\frac{a}{V}\right)^e \left(\frac{\mu}{V\rho\sqrt{S}}\right)^f \tag{8.6.4}$$

This method has now been completed. Note that the derived equation is equal to equation 8.4.15, and that the formula can therefore still be changed in an identical way as was performed after equation 8.4.15. However, that is not part of this method, and we will leave it to this.

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