Aircraft Performance 2 Summary

1. Cruise Flight

1.1 Equations of Motion

Flying airplanes spend most of the time in cruise flight. It is therefore an important part of the flight. To learn more about the cruise flight, we look at the forces acting on the aircraft during a steady symmetrical flight. These forces, the **thrust** T, the **drag** D, the **weight** W and the **lift** L are shown in figure 1.1.

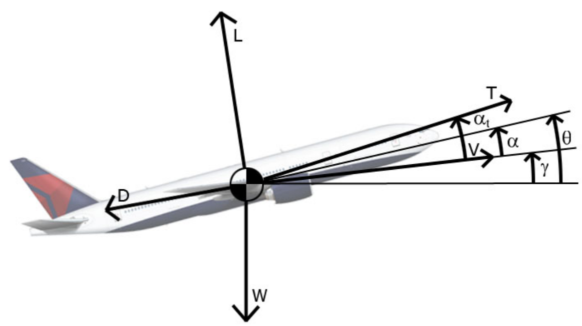

Figure 1.1: Forces acting on the aircraft.

Per definition the weight points downward, the lift is perpendicular to the velocity vector and the drag is parallel to the velocity vector. We also assume that the thrust points in the direction of the velocity, so the **thrust angle of attack** $\alpha_t = 0$. Other angles are the **flight path angle** γ, the **angle of attack** α and the **pitch angle** θ.

We now use Newton's second law, $\mathbf{F} = m\mathbf{a}$. Looking in the direction parallel to the velocity will give

$$\frac{W}{g}\frac{dV}{dt} = T - D - W\sin\gamma. \tag{1.1.1}$$

Looking in the direction perpendicular to the velocity will give

$$\frac{W}{g}V\frac{d\gamma}{dt} = L - W\cos\gamma. \tag{1.1.2}$$

Furthermore, we have two kinematic equations, being

$$\frac{ds}{dt} = V\cos\gamma \qquad \text{and} \qquad \frac{dH}{dt} = V\sin\gamma. \tag{1.1.3}$$

A cruise flight is a **quasi-stationary flight**, meaning that at any given point you can assume the flight is stationary. However, you can not assume that the entire flight is stationary. In fact, the weight of the aircraft decreases during flight, according to

$$\frac{dW}{dt} = -F(V, H, \Gamma), \tag{1.1.4}$$

where F is the fuel flow, which depends on the velocity V, the height H and the engine setting Γ. Cruise flights are usually steady horizontal flights. **Steady flight** implies that $dV/dt = 0$. **Horizontal flight** means that $\gamma = 0$. Combining this with the equations of motion gives

$$T = D \qquad \text{and} \qquad L = W. \tag{1.1.5}$$

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1.2 Cruise Flight Strategy

During the cruise flight you want to use as few fuel as possible. If you want to stay as long in the air as possible, you should maximize the endurance. The **endurance** E is the time which you can stay in the air with a given amount of fuel. To maximize E you should minimize the fuel flow F.

If, however, you want to go as far as possible, you should maximize the range. The **range** R is the distance you can cover with a given amount of fuel. So the distance ds per weight of fuel used dW_f should be maximized. So we need to maximize

$$\frac{ds}{dW_f} = \frac{ds/dt}{dW_f/dt} = \frac{V}{F}. \tag{1.2.1}$$

For jet engines we can assume that the fuel flow varies linearly with the thrust, so $F = c_T T = c_T D$, where the constant term c_T is the **specific fuel consumption**. Also, from the definition of the lift and drag coefficient follows that $D/C_D = L/C_L = W/C_L$. Combining all this gives

$$\frac{V}{F} = \frac{V}{c_T D} = \frac{V}{c_T W}\frac{C_L}{C_D} = \sqrt{\frac{1}{SWc_T^2}\frac{2}{\rho}\frac{C_L}{C_D^2}}. \tag{1.2.2}$$

So to increase the range we should decrease ρ. That is simple. We should fly as high as possible.

Now let's consider we're flying at a given altitude (and thus constant density ρ). We need to maximize C_L/C_D^2. Differentiating this fraction by using $C_D = C_{D_0} + \frac{C_L^2}{\pi Ae}$ will show that C_L/C_D^2 is minimal if

$$C_L = \sqrt{\frac{1}{3}C_{D_0}\pi Ae}. \tag{1.2.3}$$

So we now have sufficient data to determine our strategy. We should fly as high as possible and choose our V such that V/F is at a maximum.

But what if the determined velocity V with which we need to fly is beyond a **speed limit** of our aircraft? In that case we should fly at the speed limit, so $V = V_{lim}$. It now is important to minimize the drag D. Minimum drag occurs if C_L/C_D is maximal, resulting in a lift coefficient of $C_L = \sqrt{C_{D_0}\pi Ae}$. We also need to choose our height H (and thus the density ρ) such that the drag D is minimal. This occurs if

$$\rho_{opt} = \frac{W}{S}\frac{2}{V_{lim}^2}\frac{1}{C_L}. \tag{1.2.4}$$

1.3 Finding the Range

Now we are curious, what will the range be if we fly in this optimal condition? To find an equation for the range, we first express the distance traveled ds as a function of the change in weight dW according to

$$F = -\frac{dW}{dt} = -\frac{dW}{ds}\frac{ds}{dt} = -\frac{dW}{ds}V \qquad \Rightarrow \qquad ds = -\frac{V}{F}dW. \tag{1.3.1}$$

The range can now be found by integration. To get a meaningful result, we make a change-of-variable according to the above relation. So the range is

$$R = \int_{s_0}^{s_1} ds = \int_{W_0}^{W_1} -\frac{V}{F}dW = -\frac{V}{c_T}\frac{C_L}{C_D}\int_{W_0}^{W_1}\frac{1}{W}dW = \frac{V}{c_T}\frac{C_L}{C_D}\ln\left(\frac{W_0}{W_1}\right), \tag{1.3.2}$$

where W_0 is the initial mass of the aircraft and W_1 the final mass. The above equation is called the **equation of Breguet**. We will examine it more closely at the end of this chapter.

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1.4 Optimum Cruise Condition

We have determined how to fly as optimal as possible at a given point. But as we continue flying our weight W decreases and we're not flying optimal anymore. What should we do? We can consider several flying strategies and then select the best.

1. Keep constant engine settings Γ and height H. Although the velocity V will increase, the factor V/F will be far below optimal during the rest of the flight.
2. Keep constant velocity V and height H. This time the factor V/F will be better, but it will still be below optimal.
3. Keep constant angle of attack α and height H. Doing this will result in the optimal V/F during flight. However, the velocity V will decrease, which is unwanted if we want to get somewhere fast.
4. Keep constant angle of attack α and velocity V. The value of V/F will still remain optimal. The height at which we are flying increases during flight. This is usually considered the best strategy.

So the conclusion is that it is best to fly at a constant velocity V and angle of attack α with a slightly positive flight path angle γ.

1.5 Propeller Aircrafts

The previous paragraphs were all about jet aircrafts. Now let's briefly consider propeller aircrafts. There is a fundamental difference between these two aircrafts. For jet aircrafts the fuel consumption is approximately proportional to the thrust. For propeller aircrafts the fuel consumption is more or less proportional to the power P_{br} of the engine. So

$$F = c_p P_{br} = \frac{c_p}{\eta_p} P_a = \frac{c_p}{\eta_p} TV, \tag{1.5.1}$$

where η_p is the propulsive efficiency. The factor V/F, which is so important for the range, now becomes

$$\frac{V}{F} = \frac{\eta_p}{c_p} \frac{C_L}{C_D} \frac{1}{W}. \tag{1.5.2}$$

Deriving the equation of Breguet for propeller aircrafts will then give

$$R = \frac{\eta_p}{c_p} \frac{C_L}{C_D} \ln\left(\frac{W_0}{W_1}\right). \tag{1.5.3}$$

This looks rather similar to the equation of the range for jet aircrafts. This can be illustrated further if we look at the efficiency of both engines. The total efficiency of a propeller aircraft is

$$\eta_{tot} = \frac{P_{out}}{P_{in}} = \frac{TV}{H\frac{F}{g}} = \frac{\eta_p}{c_p} \frac{g}{H}, \tag{1.5.4}$$

where H is the **specific energy** of the fuel in J/kg. The total efficiency of a jet aircraft is

$$\eta_{tot} = \frac{P_{out}}{P_{in}} = \frac{TV}{H\frac{F}{g}} = \frac{V}{c_T} \frac{g}{H}, \tag{1.5.5}$$

Filling this in in Breguet's equation will give the same result for both types of aircraft, being

$$R = \frac{H}{g} \eta_{tot} \frac{C_L}{C_D} \ln\left(\frac{W_0}{W_1}\right). \tag{1.5.6}$$

The factor H/g is a measure of the fuel quality. The factor η_{tot} indicates the quality of the propulsion system. The factor C_L/C_D shows the aerodynamic quality. And finally the fraction W_0/W_1 indicates the quality of the structure of the aircraft. So the conclusion is simple: Well-designed aircrafts fly further with less fuel.

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2. Take-Off

2.1 The Procedure

Aircrafts usually fly. But before they can fly, they first have to take off from the ground. There are several moments during the take-off run that deserve some special attention. We now take a close look at the lift-off procedure of a multi-engined jet aircraft. The take-off procedure for other airplanes is similar.

Before take-off, the airplane taxis to the start of the runway. At that moment its velocity is $V_0 = 0$. When clearance is given by the control tower, the pilot gives the engines maximum power. At a certain velocity V_{dec}, called the **decision speed**, the pilot needs to make a decision. It is the last moment at which the airplane can still stop its take-off. In case of an engine failure it might be worth staying on ground.

If the pilot decides to continue the take-off, the airplane will soon reach the **rotation speed** V_r. At this point the pilot pulls the nose of the aircraft up, while the rear wheels still stay on the ground. For a brief moment the aircraft will have a certain constant **rotational speed** q_{rot}, until the desired pitch angle θ_{req} is reached.

A short moment later the **lift-off speed** V_{lof} is reached. This is the speed at which the wheels no longer touch the ground. However, the take-off has not yet been completed. The aircraft still needs to reach the **screen height** H_{scr}. At this height, which is usually 35 or 50 feet, the aircraft is safe from obstacles like trees or buildings. The speed of the aircraft at the screen height must be at least a certain **climb-out speed** (sometimes called **safety speed**) V_{scr} to ensure a safe climb.

2.2 Decision Speed

All the velocities that were mentioned in the previous paragraph are given to the pilot by the aircraft manual. However, some one needs to write the manual, so there must be a way to determine these velocities. We primarily examine the decision speed.

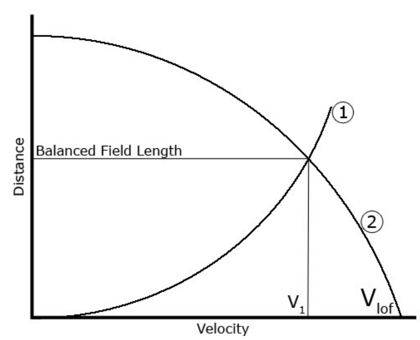

Figure 2.1: Clarification of decision speed.

Let's take a look at figure 2.1. Two graphs are displayed. Graph 1 is the distance you need to stop if you apply full brakes at the given velocity. Graph 2 is the distance you need to take off if one engine fails. If engine failure occurs at low velocities, the distance to stop is small. So it is preferable to stop. For high velocities stopping requires to much runway space. It's wiser to take off with the remaining engines.

The intermediate point, where stopping requires as much distance as taking off, is of special interest to us. The velocity at which this occurs is the **decision speed**. At this speed you can still stop the aircraft,

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but if you continue accelerating, you have to lift off. The distance you need to either stop or to take off is called the **balanced field length**.

Now let's look at the other velocities. The screen height is usually equal to the minimum velocity plus a safety factor. So $V_{scr} = jV_{min}$, where for most aircrafts $j = 1.2$. The rotation speed V_r is then set at the velocity such that V_{scr} is still reached at the screen height if engine failure occurs.

You may also be wondering what the procedure is if you only have one engine on your aircraft. If that engine fails, you should brake if you're able to. If you are not, you should still take off and hope there is an empty field ahead of you in which you can make an emergency landing.

2.3 Equations of Motion for Ground Phase

To learn more about the ground phase, we look at the forces acting on the aircraft. These forces are shown in figure 2.2.

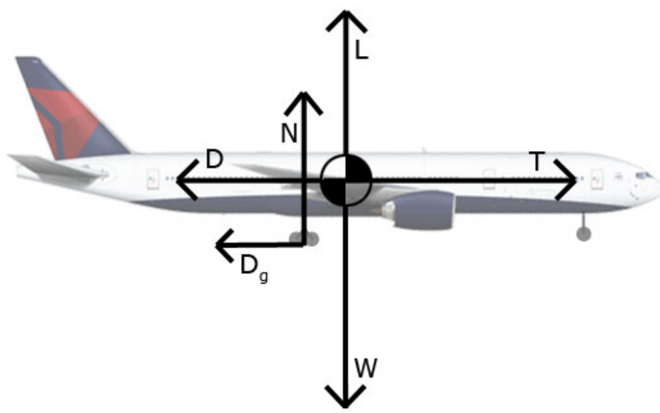

Figure 2.2: Forces acting on the aircraft during ground run.

We have seen all forces already in the chapter about cruise, except the **normal force** N and the **gear drag** D_g. But what can we say about this gear drag? We can assume that it is proportional to the normal force N, so

$$D_g = \mu N = \mu(W - L), \tag{2.3.1}$$

where μ is the friction coefficient. This coefficient depends on the wheels of the aircraft, as well as the environment. If we now look at the forces acting on the aircraft in horizontal direction, we will find

$$\frac{W}{g}\frac{dV}{dt} = T - D - D_g = T - D - \mu(W - L). \tag{2.3.2}$$

It is important to note the other assumptions we have made. We assume that the runway is horizontal, that there is no wind and once more that the thrust is in the direction of the velocity.

When you apply normal aerodynamics equations to calculate the aerodynamic forces L and D, you should be careful. Since the aircraft is still on the ground, there is the so-called **ground effect**. The air which our wings push down is pushed back up by the ground. This causes the effective angle of attack to increase. The lift therefore goes up. The ground also reflects the vortices our wing creates, partially canceling them. This causes the induced drag to decrease. The ground effect is therefore considered to be a positive effect for airplanes.

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2.4 Take-Off Distance

Now we are curious about the amount of runway the aircraft needs for the entire take-off procedure. We first look at the ground phase. From dynamics we know that $a\,ds = V\,dV$. Rewriting gives

$$s_{ground} = \int_0^{V_{lof}} \frac{V\,dV}{a} = \frac{1}{\bar{a}} \int_0^{V_{lof}} V\,dV = \frac{1}{2\bar{a}} V_{lof}^2. \quad (2.4.1)$$

Here the quantity $\bar{a}$ is the **mean acceleration**, which is equal to

$$\bar{a} = \frac{g}{W}\left(\bar{T} - \bar{D} - \bar{D}_g\right), \quad (2.4.2)$$

where $\bar{T}$, $\bar{D}$ and $\bar{D}_g$ are the mean thrust, drag and gear drag, respectively. We use this mean acceleration because it is very hard to express the thrust, drag and gear drag as a function of the velocity during a ground run. In practice, however, the mean acceleration $\bar{a}$ is approximately equal to the acceleration at $\frac{1}{2}\sqrt{2}V_{lof}$. So the thrust, drag and gear drag only have to be calculated at one given velocity to find the take-off distance corresponding to the ground phase.

Now let's consider the airborne part of the take-off. To find this, we first have to rewrite the equation of motion to

$$T - D - W\sin\gamma = \frac{W}{g}\frac{dV}{dt} = \frac{W}{g}\frac{dV}{ds}\frac{ds}{dt} = \frac{W}{g}\frac{dV}{ds}V = \frac{W}{2g}\frac{d(V^2)}{ds}. \quad (2.4.3)$$

Rewriting again will now give the so-called **equation of energy**, being

$$(T - D)ds = \frac{W}{2g}d(V^2) + W\sin\gamma\,ds = \frac{W}{2g}d(V^2) + W dH, \quad (2.4.4)$$

where the variable H indicates the height. The left side of this equation indicates the power that is put into the system, while the right side shows the change in kinetic and potential energy. To find the distance needed to reach the screen height, we once more integrate. We will find

$$\int_0^{s_{air}} (T - D)ds = \int_{V_{lof}}^{V_{scr}} \frac{W}{2g}d(V^2) + \int_0^{H_{scr}} W dH, \quad (2.4.5)$$

$$(\bar{T} - \bar{D})s_{air} = \frac{W}{2g}\left(V_{scr} - V_{lof}\right) + W H_{scr}. \quad (2.4.6)$$

At the screen height we should have reached a steady climb. We therefore may assume that $(\bar{T} - \bar{D}) = (T-D)_{scr}$. Since the speed is constant in a steady climb, we know that $dV_{scr} = 0$ and thus also $dV_{scr}^2 = 0$. This implies that $(T - D)_{scr} - W\sin\gamma_{scr} = 0$, or equivalently $W\gamma_{scr} = (T - D)_{scr} = (\bar{T} - \bar{D})$, where we have used $\sin\gamma_{scr} \approx \gamma_{scr}$. Now we find that

$$\gamma_{scr} s_{air} = \frac{V_{scr}^2 - V_{lof}^2}{2g} + H_{scr}. \quad (2.4.7)$$

To find the total take-off distance, simply add up the airborne distance s_{air} to the ground run distance s_{ground}.

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3. Climbing Flight

3.1 Rate of Climb

In the previous chapter we saw the energy equation. If we differentiate it with respect to the time t we will find

$$(T-D)\,\frac{ds}{dt} = \frac{W}{2g}\frac{d(V^2)}{dH}\frac{dH}{dt} + W\frac{dH}{dt}. \tag{3.1.1}$$

Note that we have made a change of variable to the height H in the middle term. We did this because we wanted to express things as a function of the **rate of climb**, denoted by $RC = dH/dt$. We can thus rewrite this as

$$\frac{T-D}{W}V = \frac{1}{2g}\frac{d(V^2)}{dH}RC + RC. \tag{3.1.2}$$

Let's briefly consider stationary flight. In this case $d(V^2) = 0$, so the so-called **stationary rate of climb**, denoted by RC_{st}, will be

$$RC_{st} = \frac{T-D}{W}V = \frac{Pa-Pr}{W} = P_s, \tag{3.1.3}$$

where P_s is the **specific excess power**. It is the excess power per Newton of aircraft weight. Combining this with the previous equation will

$$\frac{RC_{st}}{RC} = 1 + \frac{1}{2g}\frac{d(V^2)}{dH}. \tag{3.1.4}$$

The factor on the right hand side of this equation is called the **kinetic correction factor**. The reason for this is in fact quite logical. Your engines produce a certain amount of excess power $(T-D)\,V$. If V is constant, this energy is spend on increasing the potential energy (that is, climbing). If V increases, the kinetic energy increases, so less energy can be spend on increasing the potential energy. A correction factor is then necessary to indicate the difference between the stationary and non-stationary rate of climb.

3.2 Climb Time

The **climb time** is the time it takes to reach a given height. It can be found by using the definition of the rate of climb. The result will be

$$t_{climb} = \int_0^{t_{climb}} dt = \int_0^H \frac{1}{RC}dH. \tag{3.2.1}$$

Let's assume we want to minimize climb time for a small propeller aircraft. To do this, we need to maximize the rate of climb. Since a propeller aircraft is a low speed aircraft, the quantity $d(V^2)/2 = V\,dV$ is small enough to be neglected. The kinetic correction factor then becomes 1, so we can assume $RC = RC_{st} = P_s$. So we need to maximize

$$P_s = \frac{Pa-Pr}{W} = \frac{Pa}{W} - \frac{DV}{L} = \frac{Pa}{W} - \frac{C_D V}{C_L} = \frac{Pa}{W} - \sqrt{\frac{2}{\rho}\frac{W}{S}\frac{C_D^2}{C_L^3}}. \tag{3.2.2}$$

Since propeller aircrafts have a constant power available, the left side of the equation is constant. We therefore need to minimize C_D^2/C_L^3. The resulting lift coefficient will be $C_L = \sqrt{3C_{D_0}\pi Ae}$. The corresponding true airspeed V and equivalent airspeed V_e will be

$$V = \sqrt{\frac{2}{\rho}\frac{1}{C_L}\frac{W}{S}} \qquad \text{and} \qquad V_e = V\sqrt{\frac{\rho}{\rho_0}} = \sqrt{\frac{2}{\rho_0}\frac{1}{C_L}\frac{W}{S}}. \tag{3.2.3}$$

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Here we see that the optimal equivalent airspeed remains constant for different heights. In fact, when pilots in real low-speed aircrafts climb, they usually keep a constant indicated airspeed.

But what about high speed aircrafts? In this case the climb time becomes

$$t_{climb} = \int_0^H \frac{1}{RC} dH. = \int_0^H \frac{1 + \frac{V}{g_0}\frac{dV}{dH}}{RC_{st}} dH. \tag{3.2.4}$$

To maximize the climb time is very difficult now. If we maximize the rate of climb at one given time, we will get a very low value at another time. It is therefore important to maximize the rate of climb over the entire climb. But how to do that is a question we're currently not able to answer.

3.3 Zoom Flight and Energy Height

When an aircraft is flying, it has both kinetic energy and potential energy. These energies can be easily converted to each other by initiating a dive, or by pulling the aircraft's nose up. A flight procedure where one type of energy is converted to the other is called a **zoom flight**. To indicate the effect of zoom flights, we look at the **energy** E of the airplane, which stays constant during a zoom flight. This is

$$E = WH + \frac{W}{2g}V^2. \tag{3.3.1}$$

If we define the **specific energy** E_h as the energy per Newton of weight, we will find

$$E_h = \frac{E}{W} = H + \frac{1}{2g}V^2. \tag{3.3.2}$$

We see that E_h is more or less similar to the height. In fact, if the airplane converts all its kinetic energy to potential energy, then its height H will be equal to the specific energy E_h of the aircraft. The specific energy E_h is therefore also called the **energy height**. We now look at how this energy height changes in time. This goes according to

$$\frac{dE_h}{dt} = \frac{dH}{dt} + \frac{1}{2g}\frac{d(V^2)}{dt} = V \sin\gamma + \frac{V}{g}\frac{dV}{dt}. \tag{3.3.3}$$

We can rewrite the equation of motion of the aircraft to

$$\frac{1}{g}\frac{dV}{dt} = \frac{T - D}{W} - \sin\gamma. \tag{3.3.4}$$

Combining the previous two equations will result in a rather simple relation. This is

$$\frac{dE_h}{dt} = \frac{T - D}{W}V = P_s. \tag{3.3.5}$$

So the time derivative of the specific energy is the specific excess power. Let's suppose we want to reach a certain energy height with our aircraft. How long would that take us? This is suddenly a rather easy problem. The time it takes will be

$$t_{climb} = \int_0^{H_e} \frac{1}{P_s} dH_e. \tag{3.3.6}$$

So the conclusion is simple. To minimize the climb time, we should maximize the specific excess power for every H. Especially supersonic aircrafts make good use of this fact. When they are climbing they will, at some point, reach the transonic barrier. At transonic flight speeds the drag D is very high, resulting in a low specific excess power. To minimize this effect, a supersonic aircraft initiates a dive as soon as he reaches the transonic barrier. In this way he spends as few time as possible in the transonic flight regime. So to climb as efficient as possible, a dive is necessary. Although this is rather counterintuitive, it has proven to work in practice.

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4. Landing

4.1 The Procedure

It's nice to get an airplane up in the air. But what goes up must come down, and preferably as smooth as possible. Therefore a landing is a part of every flight.

Before an airplane can land, it first must descend. It does this with a certain **descend angle** $\bar{\gamma}$, defined as $\bar{\gamma} = -\gamma$, as is be shown in figure 4.1. This angle is usually $\bar{\gamma} = 3°$. The **approach velocity** should minimally be at $V_a = 1.3V_{min}$, in case of any unexpected events. The flight path and approach velocity are controlled by the pilot. If the airplane is going too fast, the pilot will pull up the nose of the aircraft. If the descend angle is wrong, the pilot will change the power setting of the engine.

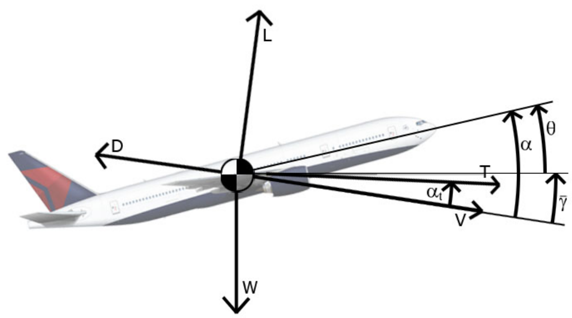

Figure 4.1: Forces acting on the aircraft during descend.

The official landing procedure starts when the aircraft reaches the screen height, which is about 50 feet for landings. When the aircraft almost hits the ground, the pilot will turn off the power on the engines and the aircraft will **flare**. The flare is the maneuver in which the pilot pulls up the nose of the aircraft, decreasing the rate of descent drastically.

When the aircraft makes a **touchdown** on the ground, it should have a **touchdown velocity** of about $V_t = 1.15V_{min}$. After that it will hit the brakes. Depending on the aircraft, several other things to slow down the aircraft can be used. The engines can switch to reverse thrust. Also spoilers can be used to reduce lift. Both mechanisms will be discussed later in this chapter.

4.2 Energy Landing Equation

Just like in the landing, we would like to know the distance an aircraft needs to land. First we look at the part of the landing in which the aircraft is still in the air. This part starts when the aircraft is at the screen height. To find an equation for the distance, we first have to look at the equation of motion. For a landing aircraft, this is

$$\frac{W}{g}\frac{dV}{dt} = T - D + W\sin\bar{\gamma}. \tag{4.2.1}$$

Let's divide this equation by the weight first. If we then multiply it by dt and multiply again $V = ds/dt$ we will get

$$\frac{1}{2g}d(V^2) = \frac{T-D}{W}ds + \sin\bar{\gamma}\,ds = \frac{T-D}{W}ds - dH. \tag{4.2.2}$$

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Now we can integrate from the screen height (with height H_{scr} and approach velocity V_a) to the ground (with height 0 and touchdown velocity V_t). The result will be

$$\frac{V_t^2}{2g} - \left(\frac{V_a^2}{2g} + H_{scr}\right) = \frac{\bar{T} - \bar{D}}{W} s_{air} \tag{4.2.3}$$

Just like we did in previous parts, we have used the mean thrust $\bar{T}$ and drag $\bar{D}$. This time we assume that the value of $\bar{T} - \bar{D}$ is the average of the value at screen height $(\bar{T} - \bar{D})_{scr}$ and the value at touchdown $(\bar{T} - \bar{D})_t$. At screen height the aircraft is in a steady descent, so $(\bar{T} - \bar{D})_{scr} = W\gamma = -W\bar{\gamma}_a$, where $\bar{\gamma}_a$ is the approach angle. At touchdown the pilot will have put out the engines, so $T_t = 0$. Therefore $(\bar{T} - \bar{D})_t = -D_t = -W\left(\frac{C_D}{C_L}\right)_t$. Combining all the data gives

$$\frac{V_t^2}{2g} - \left(\frac{V_a^2}{2g} + H_{scr}\right) = \frac{\bar{T} - \bar{D}}{W} s_{air} = -\frac{1}{2} s_{air} \left(\bar{\gamma}_a + \left(\frac{C_D}{C_L}\right)_t\right). \tag{4.2.4}$$

This equation is called the **energy landing equation**. This is because once more energy is involved. The left hand side of the equation is proportional to the change in energy, while the middle side is related to the work done on the aircraft. Using the energy landing equation, the value of s_{air} can be determined.

4.3 Brake Distance

When the aircraft safely touches down on the ground, how much distance does it need to brake? To find this distance, we once more use the equation $a\,ds = V\,dV$. The **brake distance** then is

$$s_{ground} = \int_{V_t}^{0} \frac{V\,dV}{a} = -\frac{V_t^2}{2\bar{a}}. \tag{4.3.1}$$

Note that we have again used the mean acceleration. The minus sign on the right hand side is present because this mean acceleration is negative. (The aircraft is slowing down.) In fact, it can be expressed as

$$\bar{a} = -\frac{g}{W}\left(\bar{D} + \bar{D}_g - \bar{T}\right) = -\frac{g}{W}\left(\bar{D} + \bar{D}_g + \bar{T}_{rev}\right), \tag{4.3.2}$$

where T_{rev} is the magnitude of the **reverse thrust**. Just like during take-off, it is a good approximation to take values of D, D_g and T_{rev} at a velocity of $\frac{1}{2}\sqrt{2}V_t$ as the mean values.

Substituting the previous equation in the one before it will give the brake distance. It will be

$$s_{ground} = \frac{W}{g} \frac{V_t^2}{\bar{D} + \bar{D}_g + \bar{T}_{rev}}. \tag{4.3.3}$$

4.4 Braking

For the aircraft to come to a full stop, brakes are used. It is useful to know how those brakes work. That is why the wheels of an aircraft are tested very well before they are used. To do this, engineers put them on a floor which is able to move. Let's suppose the floor will move with a velocity V, as is shown in figure 4.2. The wheel with radius r will then get an angular velocity ω. If the wheel does not slip, then $V = \omega r$. If the wheel does slip, these two values are not equal. The so-called **slip** is therefore defined as the dimensionless factor

$$\text{slip} = \frac{V - \omega r}{V}. \tag{4.4.1}$$

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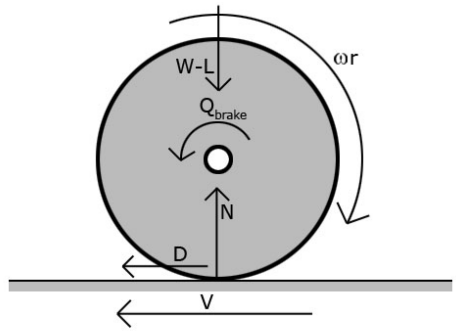

Figure 4.2: Forces/Moments acting on the wheel of an aircraft.

Now let's examine the forces acting on the wheel. From examining vertical forces we find that $N = W - L$. There is only one horizontal force, being the ground drag. This ground drag is equal to

$$D_g = \mu N = \mu (W - L), \tag{4.4.2}$$

where μ is the friction coefficient of the wheel. We want to come to a stop as fast as possible, so we intent to maximize the ground drag. We do this by maximizing the friction coefficient. It turns out that μ highly depends on the slip. For small slip values the friction coefficient still increases. However, at a certain point the coefficient reaches its maximum μ_{max}. If the slip increases even further, μ decreases again. So it's wise to allow some slip, but not too much.

If we now examine the moments acting on the aircraft, we will find that

$$\mu (W - L) r - Q_{brake} = I \frac{d\omega}{dt}, \tag{4.4.3}$$

where Q_{brake} is the **brake torque** applied by the braking system. This brake torque can be set. Given a certain brake torque, equilibrium points can be found for ω. Knowing ω, also the slip can be determined. However, some of these equilibrium points are stable, while others are unstable. A **stable** equilibrium point means that ω will return to its original value after small deviations. Stable equilibrium points have a positive value of $d\mu/d(\text{slip})$.

For **unstable** equilibrium points the opposite happens. When small deviations occur, the slip diverges away from the equilibrium point. This could result in a rapid increase in slip and thus a rapid decrease in ground drag. To prevent this, aircrafts are often equipped with an **anti-lock braking system** (ABS). Unstable equilibrium points have a negative value of $d\mu/d(\text{slip})$.

To increase the ground drag, you don't only need a high friction coefficient. You also need a normal force. When the aircraft touches down, the lift is still approximately equal to the weight, so $N = W - L = 0$. To solve this problem, **spoilers** are used. These are small deflectable plates on top of the wing. If they are deployed, they disturb the airflow and decrease the lift enormously.

So the landing procedure can now be determined. As soon as you touch down, deploy the spoilers and brake in such a way that the friction coefficient is as high as possible.

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5. Coordinate Systems

5.1 System Definitions

In the previous chapters we have derived equations of motion in a rather simple way: By drawing a picture and examining forces. This gets increasingly complicated if the flight is more three-dimensional. That's why it is useful to define various coordinate systems.

We start with the **geodetic system**, which has subscript g. The x_g-axis is defined as pointing north, while the z_g-axis points downward, to the center of the earth. The y_g-axis can now be found by using the right-hand rule. The origin of the coordinate system is at any point on the surface of the earth.

Now let's shift the origin of the geodetic system to the center of gravity (COG) of the aircraft. Now we have found the **moving earth system**, having subscript e.

Now we the road splits up in two paths. We could rotate the coordinate system to the **body system** of the aircraft (subscript b). The x_b-axis then coincides with the longitudinal axis of the aircraft (forward direction being positive). The y_b axis coincides with the latitudinal axis of the aircraft. Its positive direction is such that the z_b-axis (found with the right-hand rule) points downward.

The body system isn't very useful in performance. That's why we'll mostly use the **air path system** (subscript a). Now we rotate the coordinate system such that the x_a-axis points in the direction of the velocity. Just like in the body system, the z_a-axis points downward.

5.2 Equations of Motion for Symmetric Flight

To demonstrate the use of these coordinate systems, we will derive the equations of motion for a symmetric flight. Let's call E the matrix $\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\end{bmatrix}^T$. The position of any point in the air path system can be expressed as

$$\mathbf{r} = \begin{bmatrix} x & y & z \end{bmatrix} E_a, \tag{5.2.1}$$

where $E_a = \begin{bmatrix}\mathbf{i_a} & \mathbf{j_a} & \mathbf{k_a}\end{bmatrix}^T$. The velocity can then be expressed as

$$\mathbf{V} = \dot{\mathbf{r}} = \begin{bmatrix} V & 0 & 0 \end{bmatrix} E_a. \tag{5.2.2}$$

Now we see why it is convenient to use the air path system. Since the x_a-axis points per definition in the direction of the velocity vector, we have an easy relation here. We can once more take a time derivative to find the acceleration. This is

$$\mathbf{a} = \dot{\mathbf{V}} = \begin{bmatrix} \dot{V} & 0 & 0 \end{bmatrix} E_a + \begin{bmatrix} V & 0 & 0 \end{bmatrix} \dot{E}_a. \tag{5.2.3}$$

Note that we had to use the product rule. But what is $\dot{E}_a$? We know that the components of E_a can not change in magnitude, as they are unit vectors. They can only change in direction, when something rotates. We have assumed we're in a symmetric flight, so the only rotation is when γ changes. Expressing $\dot{E}_a$ in $\dot{\gamma}$ gives

$$E_a = \begin{bmatrix} -\mathbf{k}\dot{\gamma} & 0 & \mathbf{i}\dot{\gamma} \end{bmatrix}. \tag{5.2.4}$$

The **rotation operator** R is defined such that

$$\dot{E}_a = \begin{bmatrix} 0 & 0 & -\dot{\gamma} \\ 0 & 0 & 0 \\ \dot{\gamma} & 0 & 0 \end{bmatrix} E_a = R\,E_a. \tag{5.2.5}$$

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If we insert this in the relation for **a**, we will find

$$\mathbf{a} = \begin{bmatrix}\dot{V} & 0 & 0\end{bmatrix} E_a + \begin{bmatrix}V & 0 & 0\end{bmatrix} R E_a = \begin{bmatrix}\dot{V} & 0 & 0\end{bmatrix} E_a + \begin{bmatrix}0 & 0 & -V\dot{\gamma}\end{bmatrix} E_a = \begin{bmatrix}\dot{V} & 0 & -V\dot{\gamma}\end{bmatrix} E_a. \quad (5.2.6)$$

We can also express the forces in the air path system. We once more assume the thrust is in the direction of the velocity. Working everything out gives

$$\mathbf{F} = \begin{bmatrix}T - D - W\sin\gamma & 0 & L - W\cos\gamma\end{bmatrix} E_a. \quad (5.2.7)$$

We can now use $\mathbf{F} = m\mathbf{a}$. This relation must hold for all three components of these vectors. Looking at the y-component will give $0 = 0$, which isn't altogether interesting. Looking at the x and z-components however, will give two interesting relations, being

$$m\dot{V} = T - D - W\sin\gamma, \qquad \text{and} \qquad mV\dot{\gamma} = L - W\cos\gamma. \quad (5.2.8)$$

These are exactly the equations of motion we already know. So this method actually works. It may seem like a lot of work now, but for difficult situations this approach is actually a lot easier than what we did in the previous chapters.

5.3 Switching Between Systems

Suppose we have certain coordinates in the moving earth system and want to transform them to the air path system. How do we do this? The key to answering this question, is the **transformation matrix**.

We go from the moving earth system to the air path system in several steps. First we rotate the moving earth system about its z-axis, such that the x-axis points in the direction of the velocity vector as much as possible. (The y-component of $\mathbf{V}$ should then be 0.) We now have rotated over the so-called **azimuth angle** χ. We call this intermediate system E_1. We can now show that

$$\begin{bmatrix}x & y & z\end{bmatrix} E_1 = \begin{bmatrix}x\cos\chi - y\sin\chi & x\sin\chi + y\cos\chi & 1\end{bmatrix} E_e = \begin{bmatrix}x & y & z\end{bmatrix} \begin{bmatrix}\cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1\end{bmatrix} E_e. \quad (5.3.1)$$

The matrix on the right-hand side (the one with the sines and cosines) is the transformation matrix, denoted by $[\chi]$. It transforms E_e to E_1. Note that we could have removed the matrix with the x, y and z in the above equations. The equation would then simply be $E_1 = [\chi] E_e$.

But we haven't reached the air path system yet. To continue, we rotate system 1 around the y-axis over the **flight path angle** γ, such that the x-axis actually coincides with the velocity vector $\mathbf{V}$. We give this intermediate system number 2. We now have

$$E_2 = [\gamma] E_1 = [\gamma] [\chi] E_e, \quad (5.3.2)$$

where the transformation matrix $[\gamma]$ is derived similar to $[\chi]$. We're almost at the air path system. There's only one remaining thing that we have neglected in all previous chapters. Previously we have only considered symmetric flights. But now we consider any type of flight. So we still need to rotate around the x-axis by the **aerodynamic roll angle** μ. The result will be

$$E_a = [\mu] E_2 = [\mu] [\gamma] [\chi] E_e. \quad (5.3.3)$$

We have now expressed the air path system as a function of the moving earth system. It is very important to note the order of the transformation matrices. Mixing up the order of the matrices will give wrong results.

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But what if we want to reverse the transformation - go from the air path system to the moving earth system? That is relatively simple. All the transformation matrices are orthonormal matrices (meaning that their columns are linearly independent and all have unit length). Any orthonormal matrix A has the property $A^{-1} = A^T$. Using this fact, we can write

$$E_e = ([\mu]\,[\gamma]\,[\chi])^{-1}\,E_a = [\chi]^{-1}\,[\gamma]^{-1}\,[\mu]^{-1}\,E_a = [\chi]^T\,[\gamma]^T\,[\mu]^T\,E_a. \tag{5.3.4}$$

Note that now also the order of the matrices has reversed.

5.4 General Equations of Aircraft Motion

We have derived the equations of motion for symmetric flights previously in this chapter. We can now derive the general equations of motion for any type of flight. First we need to consider the acceleration of the aircraft. This is still given by equation 5.2.3. However, the value of $\dot{E}_a$ has been drastically changed. By examining the aircraft motion, we can find that

$$\dot{E}_a = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix} E_a = R\,E_a, \tag{5.4.1}$$

where p is the rotational velocity about the x-axis, q is the rotational velocity about the y-axis and r is the rotational velocity about the z-axis.

Now we know the acceleration of the aircraft, we need to look at the forces acting on the aircraft. These are

$$\mathbf{F} = \mathbf{T} + \mathbf{D} + \mathbf{L} + \mathbf{W} = \begin{bmatrix} T - D & 0 & -L \end{bmatrix} E_a + \begin{bmatrix} 0 & 0 & W \end{bmatrix} E_e. \tag{5.4.2}$$

Now we see why we need the transformation matrices. To be able to work with these equations, we need to express them in the same coordinate system. It's wisest to use E_a as the coordinate system. Luckily we know how to convert something from E_e to E_a. If we work everything out and use $\mathbf{F} = m\mathbf{a}$ we will get

$$\begin{bmatrix} T - D & 0 & -L \end{bmatrix} E_a + \begin{bmatrix} 0 & 0 & W \end{bmatrix} [\chi]^T\,[\gamma]^T\,[\mu]^T\,E_a = \begin{bmatrix} m\dot{V} & 0 & 0 \end{bmatrix} E_a + \begin{bmatrix} mV & 0 & 0 \end{bmatrix} R\,E_a. \tag{5.4.3}$$

Now we can derive the general equations of motion. By examining all three components, we find three equation. These are

$$T - D - W \sin\gamma = m\dot{V}, \tag{5.4.4}$$

$$W \cos\gamma \sin\mu = mVr, \tag{5.4.5}$$

$$L - W \cos\gamma \cos\mu = mVq. \tag{5.4.6}$$

We can use transformation matrices to rewrite these equations to a more useful form, being

$$m\dot{V} = T - D - W \sin\gamma, \tag{5.4.7}$$

$$mV\dot{\chi}\cos\gamma = L \sin\mu, \tag{5.4.8}$$

$$mV\dot{\gamma} = L \cos\mu - W \cos\gamma. \tag{5.4.9}$$

These equations are useful when describing many flight types. However, they do have their constraints. They don't take into account forces/moments caused by the elevators, the ailerons or the rudder. So keep that in mind when applying them.

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6. Effects of Wind

6.1 Equations of Motion

An aircraft flies in air. So its speed is also measured with respect to the air. In case of winds, this velocity is not equal to the velocity of the aircraft with respect to the ground. Instead, this velocity can be expressed as

$$\mathbf{V_g} = \mathbf{V} + \mathbf{V_w}, \tag{6.1.1}$$

where $\mathbf{V_w}$ is the velocity of the wind with respect to the ground. We usually assume this velocity is directed parallel to the ground, in such a way that the aircraft experiences headwind. Therefore

$$\mathbf{V_w} = \begin{bmatrix} V_w & 0 & 0 \end{bmatrix} E_g. \tag{6.1.2}$$

To derive the equations of motion, we look at the acceleration of the aircraft, which is

$$\mathbf{a} = \dot{\mathbf{V}}_\mathbf{g} = \dot{\mathbf{V}} + \dot{\mathbf{V}}_\mathbf{w} = \begin{bmatrix} \dot{V} & 0 & 0 \end{bmatrix} E_a + \begin{bmatrix} V & 0 & 0 \end{bmatrix} \dot{E}_a + \begin{bmatrix} -\dot{V}_w & 0 & 0 \end{bmatrix} E_g. \tag{6.1.3}$$

Note that we have not written the term involving $\dot{E}_g$. This is because E_g doesn't change, so $\dot{E}_g = 0$. If we assume the aircraft is in a symmetric flight, we can rewrite the acceleration as

$$\mathbf{a} = \begin{bmatrix} \dot{V} - \dot{V}_w \cos\gamma & 0 & -V\dot{\gamma} - \dot{V}_w \sin\gamma \end{bmatrix} E_a. \tag{6.1.4}$$

Using Newton's second law this time results in

$$T - D - W\sin\gamma = \frac{W}{g}\left(\dot{V} - \dot{V}_w \cos\gamma\right), \tag{6.1.5}$$

$$L - W\cos\gamma = \frac{W}{g}\left(V\dot{\gamma} + \dot{V}_w \sin\gamma\right). \tag{6.1.6}$$

These are the equations of motion if the airplane experiences headwind. But not only the equations of motion are slightly different. Also the kinematics equations have changed. They now are

$$\dot{H} = V\sin\gamma \qquad \text{and} \qquad \dot{s} = V\cos\gamma - V_w. \tag{6.1.7}$$

6.2 Taking Off and Landing

When the airplane is in the air, it's relatively simple to calculate the effects of it. But what if the aircraft is near the ground? We first consider the take-off and then look at the landing.

During take-off the aircraft needs to reach the lift-off speed V_{lof} to be able to lift off from the ground. This speed is measured with respect to the wind. The speed of the aircraft is, however, measured with respect to the ground. So there is a difference. Let's suppose $V_{lof} = 100kts$. If there is a headwind of $20kts$, then the aircraft only needs to have a velocity of $V = 80kts$ with respect to the ground to take off. If, however, the wind comes from the back of the aircraft, it needs a velocity of $V = 120kts$. So if you take off with headwind, you need a much lower velocity, and thus a much shorter runway. Therefore its preferable to take off with headwinds.

The situation is virtually the same for landings. If you land with headwinds, you have a much lower velocity with respect to the ground, and therefore it's much easier to come to a full stop. If you land with the wind blowing in your back, then you need a much greater distance to come to a complete stop.

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6.3 Wind Gradients

When climbing or descending, the situation is approximately the same. It's better to climb/descend with headwinds. To show this, we derive an equation which involves the **wind gradient** dV_w/dH. First we rewrite $\dot{V}$ and $\dot{V}_w$ to

$$\dot{V} = \frac{dV}{dt} = \frac{dV}{dH}\frac{dH}{dt} = \frac{dV}{dH}RC \qquad \text{andidentically} \qquad \dot{V}_w = \frac{dV}{dH}RC. \tag{6.3.1}$$

Substituting this in equation 6.1.5 will give

$$\frac{W}{g}\left(\frac{dV}{dH}RC - \frac{dV_w}{dH}RC\cos\gamma\right) = T - D - W\sin\gamma. \tag{6.3.2}$$

Multiplying by V/W will then give

$$\frac{V}{g}RC\left(\frac{dV}{dH} - \frac{dV_w}{dH}\cos\gamma\right) + V\sin\gamma = \frac{TV - DV}{W} = \frac{P_a - P_r}{W} = RC_{st}. \tag{6.3.3}$$

Rewriting $V\sin\gamma = RC$ and bringing terms within brackets will give

$$RC\left(1 + \frac{V}{g}\frac{dV}{dH} - \frac{V}{g}\frac{dV_w}{dH}\right) = RC_{st}. \tag{6.3.4}$$

Climbs are usually performed at an approximately constant velocity, so we can assume that $dV/dH = 0$. Now look at the term dV_w/dH. Winds usually blow harder at higher altitudes. In case of headwinds this means that $dV_w/dH > 0$ and thus $RC > RC_{st}$. If the wind is in your back, then $dV_w/dH < 0$ and thus $RC < RC_{st}$. So to climb faster, it is better to have headwinds.

The situation is similar for descending. Since $RD = -RC$ and $RD_{st} = -RD_{st}$ we find

$$RD\left(1 + \frac{V}{g}\frac{dV}{dH} - \frac{V}{g}\frac{dV_w}{dH}\right) = RD_{st}. \tag{6.3.5}$$

This once more shows that $RD > RD_{st}$ in case you are descending with headwinds.

6.4 Microbursts

A microburst is a column of descending air, like is displayed in figure 6.1. As the air comes close to the ground, it diverges in multiple directions. It's similar to the water coming out of a tap. When the flow of water hits the sink, it moves away in all directions.

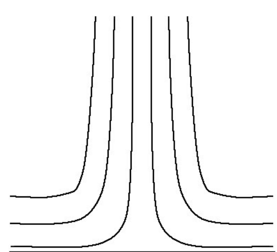

Figure 6.1: Microburst.

When an aircraft is flying into a microburst, it first experiences a headwind. It therefore slows down a bit with respect to the ground. When it passes the center of the microburst, the direction of the wind suddenly seems to reverse. The aircraft has then lost almost all of its speed, and more or less plummets to the ground. Although microbursts are relatively rare phenomena, they have caused multiple aircrafts to crash.

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7. Turning Flight

7.1 Turn Strategy

Flying in a straight line tends to get boring after a while, and it limits your possible destinations. So you need to be able to turn the aircraft. How should we do this? We consider three options

- Change μ. To see the effect, we examine the three-dimensional equations of motion that were derived previously. Let's examine the second equation. Since $\mu \neq 0$ also $\sin \mu \neq 0$ and thus $\dot{\chi} \neq 0$. So we will turn. Now look at the third equation. Since $\cos \mu$ will decrease we find that $\dot{\gamma} < 0$. That means we start plummeting to the ground. Since that is rather unwanted, this option isn't a very good one.
- In addition to changing μ also increase the angle of attack α. If we do this in such a way that $L \cos \mu$ stays constant, we won't drop to the ground. But now look at the first equation of motion. Since the angle of attack α increased also the drag D increased. Therefore $\dot{V} < 0$. This means we will slow down and reduce lift. Of course we can increase α even more, but we can't do this indefinitely. After a while we will stall and fall to the ground. So this isn't a good option either.
- Let's now change μ, increase the angle of attack α and increase the thrust T. If we increase the thrust enough to compensate for the additional drag D, we won't slow down. Therefore we are performing a perfect horizontal turn.

The third method seems to be a good one. But we have already mentioned that the equations of motion don't consider rudder and aileron deflection. In a real airplane we also need to compensate with the rudder to prevent adverse yaw. But if we do that too, we are actually able to perform a perfect horizontal turn.

7.2 Turn Radius and Load Factor

So we now know how to rotate an aircraft. But how much space do we need? From physics we know that $\mathbf{F_c} = mV^2/R$, where $\mathbf{F_c}$ is the **centrifugal force** and R is the **turn radius**. Since we are considering a horizontal turn, we know that $\gamma = \dot{\gamma} = 0$. The centrifugal force therefore is

$$F_c = L \sin \mu = \frac{W}{g} \frac{V^2}{R}. \tag{7.2.1}$$

In a horizontal turn the relation $L = W$ doesn't hold anymore. Instead, from the equations of motion we can find that

$$W = L \cos \mu = L \cos \phi, \tag{7.2.2}$$

where ϕ is a variable which is used to indicate the bank angle in horizontal turns. In this case it's equal to μ. Now we can define the **load factor** n as

$$n = \frac{L}{W} = \frac{1}{\cos \phi} = \frac{V^2}{g} \frac{1}{R \sin \mu}. \tag{7.2.3}$$

If we now use the fact that $\mu = \phi$ and also the mathematical relation $\sin^2 x + \cos^2 x = 1$ we can express the turn radius in the load factor. The result will be

$$R = \frac{V^2}{g\sqrt{n^2 - 1}}. \tag{7.2.4}$$

Note that always $n > 1$. So in every horizontal turn you make you feel slightly heavier.

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We would also like to know how long a turn takes. The angular velocity of the turn Ω is simply equal to $\Omega = V/R = \dot{\chi}$. Knowing this, we know that the time it takes to make a complete 360° turn is

$$T_{2\pi} = \frac{2\pi}{\Omega} = 2\pi\frac{R}{V} = 2\pi\frac{V}{g\sqrt{n^2-1}}. \tag{7.2.5}$$

So if you want to make a fast turn you actually have to fly as slow as possible. This may seem counterintuitive. To do something fast you need to decrease your speed. But when you're driving a bike or a car you also slow down when making a turn. So it actually makes sense to all of us.

7.3 Power During a Turn

We know that we need to increase the engine power during a turn. But by how much is this? We know that $L = nW$, so from that we can derive that

$$V = \sqrt{\frac{2}{\rho}\frac{1}{C_L}\frac{nW}{S}}. \tag{7.3.1}$$

We can also express the drag as

$$D = \frac{C_D}{C_L}nW. \tag{7.3.2}$$

If we combine these two equations, we can find the power required to be

$$P_r = DV = \sqrt{\frac{2}{\rho}\frac{C_D^2}{C_L^3}\frac{(nW)^3}{S}}. \tag{7.3.3}$$

So we find that the power required is proportional to $n\sqrt{n}$. So the higher the load factor, the higher the power required. Of course our engine has a maximum power available. So this determines the maximum load factor n_{max}. Using equation 7.3.2 and the relation $T = D$ we find that

$$n_{max} = \frac{T}{W}\frac{C_L}{C_D}. \tag{7.3.4}$$

However, for low velocities the maximum load factor isn't bounded by the power available. It is then bounded by the maximum lift coefficient $C_{L_{max}}$ according to

$$n_{max} = \frac{L_{max}}{W} = \frac{\rho V^2 S C_{L_{max}}}{2W}. \tag{7.3.5}$$

In both cases the maximum velocity in a turn will be

$$V_{n_{max}} = \sqrt{\frac{2}{\rho}\frac{1}{C_L}\frac{n_{max}W}{S}}. \tag{7.3.6}$$

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8. Helicopters

8.1 Actuator Disk Theory

In a helicopter, lift isn't created by the wings, but by rotating blades. This enables helicopters to hover. In case of hovering flight, the thrust produced by the rotor blades T must be equal to the weight W.

But how can we calculate this thrust? One way to do this is by using the **actuator disk theory**, which has already been treated in the Aerodynamics A course. We will briefly repeat that theory here. Suppose we have a set of rotating blades, called the **actuator disk**. We can consider 4 points now. Point 0 is infinitely far to the left of the disk, while point 1 is just a little bit to the left of the disk. Identically, point 2 is just to the right of the disk and point 3 is infinitely far to the right.

If we assume the velocity is evenly distributed over the actuator disk, and that the flow is incompressible and irrotational, we can derive a few simple relations. Let's define the **induced velocity** V_i as $V_i = V_1 - V_0$. We can now show that $V_2 = V_1$ and also $V_3 - V_2 = V_i$. So therefore the total velocity increment of the air is $V_3 - V_0 = 2V_i$. We can also show that the thrust now is

$$T = 2mV_i = 2\rho\pi r^2 V_i, \qquad (8.1.1)$$

where m is the mass flow of air that goes through the disk and r is the actuator disk radius.

8.2 Helicopter Power

How much power would the engine need to provide to hover? In an ideal situation the power would be the difference between the rate of power that comes out of the system, and the rate of power that goes in. This would make the so-called **ideal power**

$$P_{id} = e_{out} - e_{in} = \frac{1}{2}m\left(V_0 + 2V_i\right)^2 - \frac{1}{2}mV_0^2 = 2mV_i\left(V_0 + V_i\right) = T\left(V_0 + V_i\right). \qquad (8.2.1)$$

In a hovering flight, the initial velocity V_0, being the velocity far away from the disk, is 0. So we also know that $P_{id} = TV_i = WV_i$. If we apply $T = W$ to the last equation of the previous paragraph, we can also find that

$$V_i = \sqrt{\frac{W}{2\rho\pi r^2}}. \qquad (8.2.2)$$

The ideal power of the helicopter now becomes

$$P_{id} = WV_i = W\sqrt{\frac{W}{2\rho\pi r^2}}. \qquad (8.2.3)$$

So to reduce the ideal power, we need to decrease our weight, increase the rotor blade size or simply fly at a lower altitude. However, the ideal power isn't the actual power that is needed from the engine. There are always losses. These losses are all combined in one sign. This is the so-called **figure of merit** M, defined as

$$M = \frac{P_{id}}{P_{hov}}, \qquad (8.2.4)$$

where P_{hov} is the actual power that is necessary to hover. For most modern helicopters the figure of merit is about $M = 0.7$.

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8.3 Blade Element Theory

In the **blade element theory** we look at an infinitely small piece of a rotor blade and derive the lift dL and drag dD acting on that piece. Using those forces we can integrate to find the thrust and drag of the full rotor blades.

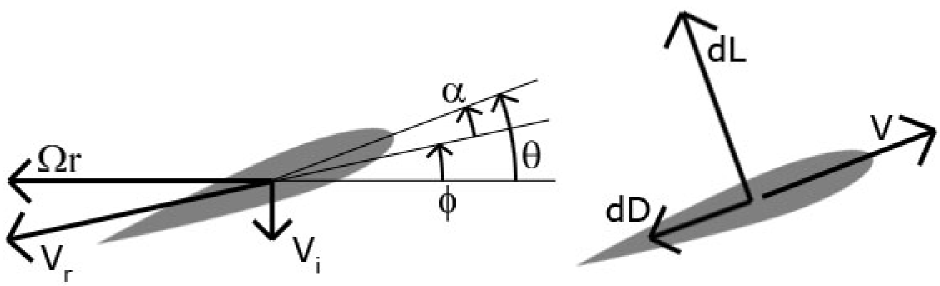

Figure 8.1: Motions and forces in the blade element theory.

So let's look at an infinitely small piece of a rotor blade, at a distance r from the center. Such a piece is shown in figure 8.1. Since the rotor rotates with an angular velocity of Ω, the small piece of rotor blade has a velocity component to the left of Ωr. Due to the induced velocity, there is a downward pointing velocity component of V_i. Together they form the resultant velocity V_r. Since this velocity is not directed horizontally, the air has a certain **inflow angle** ϕ. Also a **pitch angle** θ and an **angle of attack** α can be distinguished.

The small bit of thrust dT acting on the rotor blade part now is

$$dT = dL \cos\phi - dD \sin\phi \approx dL, \tag{8.3.1}$$

where we have approximated $\cos\phi \approx 1$ and $\sin\phi \approx 0$. The power that is needed to keep the rotor rotating is

$$dP = dD\,\Omega r = (dL \sin\phi + dD \cos\phi)\,\Omega r \approx dT\,V_i + dD\,\Omega r, \tag{8.3.2}$$

where we this time have approximated $\cos\phi \approx 1$ and $\sin\phi \approx \tan\phi = V_i/\Omega r$. To find the total power that is needed for all the rotor blades, we integrate over one entire rotor blade and multiply by the number of blades n. The power then becomes

$$P = n\int_0^R V_i\,dT + n\int_0^R \Omega r\,dD = TV_i + P_p = P_i + P_p, \tag{8.3.3}$$

where P_i is the induced component of the power and P_p is the power necessary to compensate for the profile drag of the blades. Also R is the length of the entire rotor blade. The variable P_i is usually approximated using $P_i = kP_{id}$, where the correction factor k is often about 1.2. The power component P_p is a bit more difficult to determine. If c_{d_p} is the profile drag component of the rotor blade piece, we can write

$$P_p = n\int_0^R \Omega r\,dD = n\int_0^R c_{d_p}\frac{1}{2}\rho\,(\Omega r)^3\,c dr = \frac{\bar{c}_{d_p}\sigma}{8}\left(\Omega R^2\right)^3 \pi R^2. \tag{8.3.4}$$

Here the sign $\bar{c}_{d_p}$ indicates the mean drag coefficient for the entire rotor. The factor σ is called the **rotor solidity**, defined as

$$\sigma = \frac{nRc}{\pi R^2}. \tag{8.3.5}$$

There is a simple physical meaning behind this rotor solidity. It's the part of the rotor disk that is filled with blades. The surface area of the rotor disk is simply πR^2, while the surface area of the blades is nRc.

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8.4 Forward Flight

Helicopters would tend to get boring if they can only hover. They can also fly forward. To let helicopters fly forward, you first have to rotate the rotor disk. When doing this, you have to keep in mind the gyroscopic effect, as the rotor disk is rotating quite rapidly.

The angle with which the rotor disk is tilted forward is called the **disk angle of attack** α_d. Since we're flying forward now, we experience drag. First of all a drag force H_0 acts on the rotor disk. But the rest of the helicopter (the so-called **bus**) is also suspect to drag. This is the so-called **parasite drag** D_{par}. The total power that the engine now needs to provide is

$$kTV_i + P_p + H_0 V + D_{par} V = P_i + P_p + P_{drag} + P_{par}. \tag{8.4.1}$$

Let's look at the individual power components now. To find an expression for the profile drag P_p we can once more use the blade element theory. But it's a bit more complicated this time. The velocity of a small blade element depends not only on the distance r from the rotor disk center. It also depends on the angle ψ it makes with respect to the direction of the velocity. If a small piece of rotor moves in the same direction as the velocity of the helicopter, than the two velocities need to be added up. So then $V_r = \Omega R + V \cos \alpha_d$. In the opposite case they need to be subtracted. In general we can say that

$$V_r = \Omega R + V \cos \alpha_d \sin \mu. \tag{8.4.2}$$

What we do now is simply integrate over the entire rotor disk, as if there was a rotor blade at every small angle $d\psi$. We then divide by 2π to get the average profile power for a certain rotor blade. Of course we also multiply by the number of rotor blades n. The profile power then becomes

$$P_p = \frac{n}{2\pi} \int_0^{2\pi} \left(\int_0^R c_{d_p} \frac{1}{2} \rho \left(\Omega r + V \cos \alpha_d \sin \mu \right)^2 c \, \Omega r \, dr \right) d\psi = \frac{\bar{c_{d_p}} \sigma}{8} \rho \left(\Omega R \right)^3 \pi R^2 \left(1 + \mu^2 \right). \tag{8.4.3}$$

Note that the profile power can also be written as $P_p = P_{hov} \left(1 + \mu^2\right)$, where the hover power is equal to

$$P_{hov} = \frac{\bar{c_{d_p}} \sigma}{8} \rho \left(\Omega R \right)^3 \pi R^2. \tag{8.4.4}$$

The sign μ indicates the so-called **advance ratio** and is defined as

$$\mu = \frac{V \cos \alpha_d}{\Omega R}. \tag{8.4.5}$$

In practice $\mu_{max} \approx 0.3$. If the helicopter goes any faster, two things can occur. It may occur that the blade facing headwind (the so-called **advancing blade** with velocity $\Omega R + V \cos \alpha_d$) experiences a supersonic flow. It is also possible that the so-called **retreating blade** with velocity $\Omega R - V \cos \alpha_d$ actually moves backward. Both situations are not really positive.

The drag power due to H_0 is rather difficult to determine. That's why it is simply approximated by

$$P_{drag} = H_0 V \cos \alpha_d = P_{hov} 4.65 \mu^2. \tag{8.4.6}$$

The most important contribution to the drag is the parasite drag. Especially for high velocities this drag dominates. The power necessary to counteract this drag is

$$P_{par} = D_{bus} V = C_{D_{bus}} \frac{1}{2} \rho V^3 A_{ref}, \tag{8.4.7}$$

where $C_{D_{bus}}$ is the drag coefficient of the bus and A_{ref} is a certain reference area.

Now we can draw our conclusions. The values of P_p and P_{drag} are fairly constant with increasing V. The induced power P_p even decreases if we fly forward. The parasite drag, however, strongly increases. To reduce drag it's therefore best to make the helicopter bus as aerodynamic as possible.

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8.5 Autorotation

What happens if the engine of the helicopter fails? It turns out that the helicopter will not fall to the ground. Instead, a phenomenon called **autorotation** occurs. The rotor disk starts rotating itself.

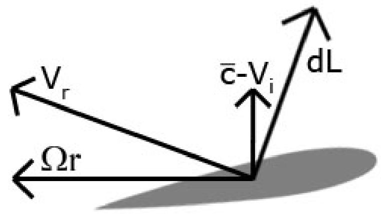

Figure 8.2: Forces acting on an autorotating rotorblade.

To explain why autorotation occurs, we examine the forces acting on the rotor blades. These forces are shown in figure 8.2. When the engine fails, the helicopter will start to lose height. It will get a certain descend speed $\bar{c}$. Due to this descend speed, it will seem like the induced velocity now has magnitude $\bar{c} - V_i$ and is pointed upward. Since the induced velocity points upward, the resultant velocity will also be tilted upward. Therefore the lift vector will be tilted in the forward direction. The vertical component of the lift vector still provides thrust, but now the horizontal component powers the rotor blades. The rotor disk rotates on its own.

This however only happens if the pilot takes action. If the engine fails, the descend speed will increase. This means that the resultant velocity vector will start to tilt upward, increasing the angle of attack. If the pilot doesn't do anything, stall will occur and almost all lift will be gone. Therefore the pilot needs to decrease the pitch angle of the rotor blade as soon as engine failure occurs.

But what if the pilot forgets to reduce the pitch? In this case the helicopter is in serious trouble. Since there is stall, the rotor blade hardly has any lift and can't power itself. The only chance the pilot has is to start performing a forward flight with the helicopter, converting potential energy to kinetic energy. When sufficient velocity has been gathered, the pilot can change that kinetic energy into rotational velocity of the rotor blade. If, however, there isn't enough height (and thus not enough potential energy) to reach the right velocity, the pilot will probably not live to learn from his mistake.

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