Calculus Formula and Data Overview

Calculus - Period 1

Differentiation and Integration

Chain Rule:

$$(f(g(x)))' = f'(g(x))g'(x) \tag{1}$$

Implicit Differentiation:
When applying implicit differentiation for a function y of x, every term with a y should, after normal differentiation (often involving the product rule), be multiplied by y' because of the chain rule. After that, the equation should be solved for y'.

Lineair Approximations:

$$f(x) - f(a) \approx f'(a)(x-a) \tag{2}$$

Mean Value Theorem:
If f is continuous on $[a,b]$ and differentiable on (a,b), then there is a c in (a,b) such that:

$$f(b) - f(a) = f'(c)(b-a) \tag{3}$$

Integration:

$$\int_a^b f(x)dx = F(b) - F(a) \tag{4}$$

Where F is any antiderivative/primitive function of f, that is, $F' = f$.

Substitution Rule:
If $u = g(x)$ then:

$$\int f(g(x))g'(x)dx = \int f(u)du \tag{5}$$

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du \tag{6}$$

Integration By Parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \tag{7}$$

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx \tag{8}$$

Improper Integrals:

$$\int_1^\infty \frac{1}{x^p}dx \tag{9}$$

This function is convergent for $p > 1$ and divergent for $p \leq 1$.

Comparison Theorem:
If $f(x) \geq g(x) \geq 0$ for $x \geq a$ then:

- If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
- If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Complex Numbers

Complex Number Notations:

$$i^2 = -1 \tag{10}$$

$$z = a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{11}$$

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta \tag{12}$$

$$\begin{gathered} |z| = r = \sqrt{a^2+b^2} \\ \theta = \arctan\tfrac{b}{a} \quad \text{or} \quad \theta = \arctan\tfrac{b}{a} + \pi \end{gathered} \tag{13}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{14}$$

Complex Number Calculation:

$$\begin{gathered} (a+bi)+(c+di) = (a+c)+(b+d)i \\ (a+bi)(c+di) = (ac-bd)+(ad+bc)i \end{gathered} \tag{15}$$

$$\begin{gathered} z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \\ r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} \end{gathered} \tag{16}$$

Complex Conjugates:

$$z = a + bi \quad \Rightarrow \quad \overline{z} = a - bi \tag{17}$$

$$\begin{gathered} \overline{z+w} = \overline{z} + \overline{w} \\ \overline{zw} = \overline{z}\,\overline{w} \\ \overline{z^n} = \overline{z}^n \\ z\overline{z} = |z|^2 \end{gathered} \tag{18}$$

Differential Equations

Separable Differential Equations:

$$\text{Form}: \ \frac{dy}{dx} = y' = P(x)Q(y)$$

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$$\text{Solution}: \int \frac{1}{Q(y)}dy = \int P(x)dx \tag{19}$$

First-Order Differential Equations

$$\text{Form}: \; y' + P(x)y = Q(x)$$

Let $\Upsilon(x)$ be any integral of $P(x)$. Solution is:

$$y = e^{-\Upsilon(x)}\left(\int e^{\Upsilon(x)}Q(x)dx + C\right) \tag{20}$$

Homogeneous second-order linear differential equations:

$$\text{Form}: \; ay'' + by' + cy = 0$$

- $b^2 - 4ac > 0$:
  Define r such that $ar^2 + br + c = 0$.
  $$\text{Solution}: \; y = c_1e^{r_1x} + c_2e^{r_2x} \tag{21}$$
- $b^2 - 4ac = 0$:
  Define r such that $ar^2 + br + c = 0$.
  $$\text{Solution}: \; y = c_1e^{rx} + c_2xe^{rx} \tag{22}$$
- $b^2 - 4ac < 0$:
  Define $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ac-b^2}}{2a}$. Solution:
  $$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \tag{23}$$

Nonhomogeneous second-order linear differential equations:

$$\text{Form}: \; ay'' + by' + cy = P(x)$$

First solve $ay_c'' + by_c' + cy_c = 0$. Then use an auxiliary equation to find one solution y_p for the given differential equation. The solution is:

$$y = y_c + y_p \tag{24}$$

Calculus - Period 2

Testing Series

Convergence/Divergence:
Suppose a is a series of numbers $a_1, a_2, \ldots$, and $s_n = \sum_{k=1}^{n} a_k$. A series s_n converges if $\lim_{n\to\infty} s_n = s$ exists as a real number. The limit s is then called the sum of series a. If s doesn't exist as a finite number, the series is divergent. Be careful not to confuse the series a_n with the series $\sum a_n = s$.

Monotonic Sequence Theorem
If a sequence is either increasing ($a_{n+1} > a_n$ for all $n \geq 1$) or decreasing ($a_{n+1} < a_n$ for all $n \geq 1$), it is called a monotonic sequence. If there are c_1 and c_2 such that $c_1 < a_n < c_2$ for all $n \geq 1$, it is called bounded. Every bounded monotonic sequence is convergent.

Test for divergence:
If $\lim_{n\to\infty} a_n$ does not exist, or if $\lim_{n\to\infty} \neq 0$, then the series s_n is divergent.

Integral test:
If f is a continuous positive decreasing function on $[1, \infty)$ and $a_n = f(n)$ for integer n, then the series s_n is convergent if, and only if, the integral $\int_1^\infty f(x)dx$ is convergent.

Comparison test:
Suppose a_n and b_n are series with positive terms and $a_n \leq b_n$ for all n, then:

- If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
- If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Limit comparison test:
Suppose a_n and b_n are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $0 < c \neq \infty$, then either both series are convergent or divergent.

Alternating series test:
If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \ldots$$

satisfies $a_{n+1} \leq a_n$ for all n and $\lim_{n\to\infty} a_n = 0$, then the series is convergent.

Absolute convergence:
A series $\sum a_n$ is called absolutely convergent if the series $\sum |a_n|$ is convergent. A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Ratio test:

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum a_n$ is absolutely convergent.

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- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then the series $\sum a_n$ is divergent.

Root test:

- If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum a_n$ is absolutely convergent.
- If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$, then the series $\sum a_n$ is divergent.

Power Series

Radius of convergence:
Power series are written as

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \tag{25}$$

where x is a variable and the c_n's are constant coefficients of the series. When tested for converges, there are only three possibilities:

- The series converges only if $x = a$. $(R = 0)$
- The series converges for all x. $(R = \infty)$
- The series converges for $|x-a| < R$ and diverges for $|x-a| > R$. For $|x-a| = R$ other means must point out whether convergence or divergence occurs.

The number R is called the radius of convergence, and can often be found using the ratio test.

Differentiation and integration:
Differentiation and integration of power functions is possible in the interval $(a-R, a+R)$, where the function does not diverge. It goes as follows:

$$\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right)' = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \tag{26}$$

$$\int \sum_{n=0}^{\infty} c_n(x-a)^n dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \tag{27}$$

Representation of functions as power series:
The first way to represent functions as power series is simple, but doesn't always work. To find the representation of $f(x)$, first find a function $g(x)$ such that $f(x) = ax^b \frac{1}{1-g(x)}$, where a and b are constants. The power series is then equal to:

$$f(x) = \sum_{n=0}^{\infty} a \cdot g(x)^{n+b} \tag{28}$$

The second way to represent functions as power series goes as follows. Let $f^{(n)}(x)$ be the n'th derivative of $f(x)$. Supposing the function $f(x)$ has a power series (this sometimes still has to be proven), the following function must be true:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{29}$$

This representation is called the Taylor series of $f(x)$ at a. For the special case that $a = 0$, it is called the Maclaurin series.

Binomial series:
If k is any real number and $|x| < 1$, the power function representation of $(1+x)^k$ is:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \tag{30}$$

$$\text{where} \quad \binom{k}{n} = \frac{k(k-1)\dots(k-n+1)}{n!} \tag{31}$$

for $n \geq 1$, and $\binom{k}{0} = 1$.

Vectors

Notation:
A vector $\mathbf{a}$ is often written as:

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{32}$$

Where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors.

Vector length:

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{33}$$

Vector addition and subtraction:

$$\mathbf{a}+\mathbf{b} = (a_x+b_x)\mathbf{i} + (a_y+b_y)\mathbf{j} + (a_z+b_z)\mathbf{k} \tag{34}$$

$$\mathbf{a}-\mathbf{b} = (a_x-b_x)\mathbf{i} + (a_y-b_y)\mathbf{j} + (a_z-b_z)\mathbf{k} \tag{35}$$

Dot product:

$$\mathbf{a}\cdot\mathbf{b} = a_xb_x + a_yb_y + a_zb_z \tag{36}$$

$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 \tag{37}$$

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \tag{38}$$

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Cross product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \quad (39)$$

$$\mathbf{a}\times\mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + \\ +(a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k} \quad (40)$$

Vector Functions:

Notation:

$$\mathbf{r}(\mathbf{t}) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad (41)$$

Differentiation and integration:

$$\mathbf{r}'(\mathbf{t}) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k} \quad (42)$$

$$\mathbf{R}(\mathbf{t}) = F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k} + \mathbf{D} \quad (43)$$

Function dependant unit vectors:

$$\mathbf{T}(\mathbf{t}) = \frac{\mathbf{r}'(\mathbf{t})}{|\mathbf{r}'(\mathbf{t})|} \quad (44)$$

$$\mathbf{N}(\mathbf{t}) = \frac{\mathbf{T}'(\mathbf{t})}{|\mathbf{T}'(\mathbf{t})|} \quad (45)$$

$$\mathbf{B}(\mathbf{t}) = \mathbf{T}(\mathbf{t}) \times \mathbf{N}(\mathbf{t}) \quad (46)$$

Trajectory length:

$$d\mathbf{s}(\mathbf{t}) = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = |\mathbf{r}'(\mathbf{t})|dt \quad (47)$$

$$s(t) = \int_a^t |\mathbf{r}'(\mathbf{t})|dt \quad (48)$$

Trajectory velocity and acceleration:

$$|\mathbf{v}(\mathbf{t})| = \frac{d\mathbf{s}(\mathbf{t})}{dt} = |\mathbf{r}'(\mathbf{t})| \quad (49)$$

$$\mathbf{a}(\mathbf{t}) = \mathbf{v}'(\mathbf{t}) = \mathbf{r}''(\mathbf{t}) \quad (50)$$

Trajectory curvature:

$$\kappa(t) = \left|\frac{d\mathbf{T}(\mathbf{t})}{d\mathbf{s}(\mathbf{t})}\right| == \frac{|\mathbf{T}'(\mathbf{t})|}{|\mathbf{r}'(\mathbf{t})|} \quad (51)$$

$$\kappa(t) = \frac{|r'(t)\times r''(t)|}{|r'(t)|^3} \quad (52)$$

Expressing acceleration in unit vectors:

$$\mathbf{a}(\mathbf{t}) = |\mathbf{v}(\mathbf{t})|'\mathbf{T}(\mathbf{t}) + \kappa|\mathbf{v}(\mathbf{t})|^2\mathbf{N}(\mathbf{t}) \quad (53)$$

$$\mathbf{a}(\mathbf{t}) = \frac{\mathbf{r}'(\mathbf{t})\cdot\mathbf{r}''(\mathbf{t})}{|\mathbf{r}'(\mathbf{t})|}\mathbf{T}(\mathbf{t}) + \frac{|\mathbf{r}'(\mathbf{t})\times\mathbf{r}''(\mathbf{t})|}{|\mathbf{r}'(\mathbf{t})|}\mathbf{N}(\mathbf{t}) \quad (54)$$

Calculus - Period 3

Functions of Multiple Variables

Definitions:
The domain D is the set (x, y) for which $f(x, y)$ exists. The range is the set of values z for which there are x, y such that $z = f(x, y)$. The level curves are the curves with equations $f(x, y) = k$ where k is a constant.

Checking for Limits:
If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$ then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist. Also f is continuous at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$

Partial Derivatives:
The partial derivative of f with respect to x at (a, b) is:

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b) \quad (55)$$

In words, to find f_x, regard y as constant and differentiate $f(x, y)$ with respect to x. f_y is defined similarly. If f_{xy} and f_{yx} are both continuous on D, then $f_{xy} = f_{yx}$.

Tangent Planes:
For points close to $z_0 = f(x_0, y_0)$ the curve of $f(x, y)$ can be approximated by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \quad (56)$$

The plane described by this equation is the plane tangent to the curve of $f(x, y)$ at (x_0, y_0).

Differentials:

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad (57)$$

If $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$ then:

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds} \quad (58)$$

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Directional Derivatives:
The directional derivative of f at (x_0, y_0) in the direction of a unit vector (meaning, $|\mathbf{u}| = 1$) $\mathbf{u} = \langle a, b\rangle$ is:

$$D_u f(x_0, y_0) = f_x(x, y)a + f_y(x, y)b = \nabla f \cdot \mathbf{u} \tag{59}$$

$$\mathbf{grad}\ f = \nabla f = \langle f_x(x, y), f_y(x, y)\rangle \tag{60}$$

The maximum value of $D_u f(x, y)$ is $|\nabla f(x, y)|$ and occurs when the vector $\mathbf{u} = \langle a, b\rangle$ has the same direction as $\nabla f(x, y)$.

Local Maxima and Minima:
If f has a local maximum or minimum at (a, b), then $f_x(a, b) = 0$ and $f_y(a, b) = 0$. If $f_x(a, b) = 0$ and $f_y(a, b) = 0$ then (a, b) is a critical point. If (a, b) is a critical point, then let D be defined as:

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2 \tag{61}$$

- If $D > 0$ then:
  - If $f_{xx}(a, b) > 0$, then $f(a, b)$ is a minimum.
  - If $f_{xx}(a, b) < 0$, then $f(a, b)$ is a maximum.
- If $D < 0$, then $f(a, b)$ is a saddle point.

Absolute Maxima and Minima:
To find the absolute maximum and minimum values of a continuous function f on a closed bounded set D, first find the values of f at the critical points of f in D. Then find the extreme values of f on the boundary of D. The largest of these values is the absolute maximum. The lowest is the minimum.

Multiple Integrals

Integrals over Rectangles:
If R is the rectangle such that $R = \{(x, y)|a \le x \le b, c \le y \le d\}$, then:

$$\iint_R f(x, y)\ dA = \int_a^b \int_c^d f(x, y)\ dy\ dx \tag{62}$$

$$\iint_R f(x, y)\ dA = \int_c^d \int_a^b f(x, y)\ dx\ dy \tag{63}$$

Integrals over Regions:
If D_1 is the region such that $D_1 = \{(x, y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}$, then:

$$\iint_{D_1} f(x, y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\ dy\ dx \tag{64}$$

If D_2 is the region such that $D_2 = \{(x, y)|a \le y \le b, h_1(y) \le x \le h_2(y)\}$, then:

$$\iint_{D_2} f(x, y)\ dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y)\ dx\ dy \tag{65}$$

Integrating over Polar Coordinates

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \tag{66}$$

$$x = r\cos\theta \qquad y = r\sin\theta \tag{67}$$

If R is the polar rectangle such that $R = \{(r, \theta)|0 \le a \le r \le b, \alpha \le \theta \le \beta\}$ where $0 \le \beta - \alpha \le 2\pi$, then:

$$\iint_R f(x, y)\ dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta \tag{68}$$

If D is the polar rectangle such that $D = \{(r, \theta)|0 \le h_1(\theta) \le r \le h_2(\theta), \alpha \le \theta \le \beta\}$ where $0 \le \beta - \alpha \le 2\pi$, then:

$$\iint_R f(x, y)\ dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta \tag{69}$$

Applications:
If m is the mass, and $\rho(x, y)$ the density, then:

$$m = \iint_D \rho(x, y)\ dA \tag{70}$$

The x-coordinate of the center of mass is:

$$\overline{x} = \frac{\iint_D x\ \rho(x, y)\ dA}{\iint_D \rho(x, y)\ dA} \tag{71}$$

The moment of inertia about the x-axis is:

$$I_x = \iint_D y^2 \rho(x, y)\ dA \tag{72}$$

The moment of inertia about the origin is:

$$I_0 = \iint_D (x^2 + y^2)\rho(x, y)\ dA = I_x + I_y \tag{73}$$

Triple Integrals
If E is the volume such that $E = \{(x, y, z)|a \le x \le b, g_1(x) \le y \le g_2(x), h_1(x, y) \le z \le h_2(x, y)\}$, then:

$$\iiint_E f(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z) dz dy dx \tag{74}$$

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Calculus - Period 4

Three-Dimensional Integrals

Cylindrical Coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \tag{75}$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z \tag{76}$$

Integrating Over Cylindrical Coordinates:

$$\int\int\int_E f(x,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \dots$$
$$\dots \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} r f(r\cos\theta, r\sin\theta, z) dz dr d\theta \tag{77}$$

Spherical Coordinates:

$$x = \rho\cos\theta\sin\phi \quad y = \rho\sin\theta\sin\phi \quad z = \rho\cos\phi \tag{78}$$

$$\rho^2 = x^2 + y^2 + z^2 \tag{79}$$

Integrating Over Spherical Coordinates:
If E is the spherical wedge given by $E = \{(\rho,\theta,\phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$, then:

$$\int\int\int_E f(x,y,z)dV = \int_a^b \int_\alpha^\beta \int_c^d \rho^2 \sin\phi \dots$$
$$\dots f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) d\rho d\theta d\phi \tag{80}$$

Change of Variables:
The Jacobian of the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{81}$$

If the Jacobian is nonzero and the transformation is one-to-one, then:

$$\iint_R f(x,y)dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \tag{82}$$

This method is similar to the one for triple integrals, for which the Jacobian has a bigger matrix and the change-of-variable equation has some more terms.

Basic Vector Field Theorems

Definitions

- A piecewise-smooth curve - A union of a finite number of smooth curves.
- A closed curve - A curve of which its terminal point coincides with its initial point.
- A simple curve - A curve that doesn't intersect itself anywhere between its endpoints.
- An open region - A region which doesn't contain any of its boundary points.
- A connected region - A region D for which any two points in D can be connected by a path that lies in D.
- A simply-connected region - A region D such that every simple closed curve in D encloses only points that are in D. It contains no holes and consists of only one piece.
- Positive orientation - The positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C.

Vector Field:
A vector field on $\mathbb{R}^n$ is a function $\mathbf{F}$ that assigns to each point (x,y) in an n-dimensional set an n-dimensional vector $\mathbf{F}(x,y)$. The gradient ∇f is defined by:

$$\nabla f(x,y,\dots) = f_x\,\mathbf{i} + f_y\,\mathbf{j} + \dots \tag{83}$$

and is called the gradient vector field. A vector field $\mathbf{F}$ is called a conservative vector field if it is the gradient of some scalar function.

Line Integrals:
The line integral of f along C is:

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{84}$$

The line integral of f along C with respect to x is:

$$\int_C f(x,y)dx = \int_a^b f(x(t),y(t)) \frac{dx}{dt}\, dt \tag{85}$$

The line integral of a vector field $\mathbf{F}$ along C is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \tag{86}$$

Where $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$ is the unit tangent vector.

Conservative Vector Fields:
If C is the curve given by $\mathbf{r}(t)$ $(a \leq t \leq b)$, then:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \tag{87}$$

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The integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.
If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a conservative vector field, then:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{88}$$

Also, if D is an open simply-connected region, and if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\mathbf{F}$ is conservative in D.

Surfaces

Parametric Surfaces:
A surface described by $\mathbf{r}(u,v)$ is called a parametric surface. $\mathbf{r_u} = \frac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{r_v} = \frac{\partial \mathbf{r}}{\partial v}$. For smooth surfaces ($\mathbf{r_u} \times \mathbf{r_v} \neq \mathbf{0}$ for every u and v) the tangent plane is the plane that contains the tangent vectors $\mathbf{r_u}$ and $\mathbf{r_v}$, and the vector $\mathbf{r_u} \times \mathbf{r_v}$ is the normal vector to the tangent plane.

Surface Areas:
For a parametric surface, the surface area is given by:

$$A = \iint_D |\mathbf{r_u} \times \mathbf{r_v}| dA \tag{89}$$

For a surface graph of $g(x,y)$, the surface area is given by:

$$A = \iint_D \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} dA \tag{90}$$

Surface Integrals:
For a parametric surface, the surface integral is given by:

$$\iint_S f(x,y,z)\, dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r_u} \times \mathbf{r_v}| dA \tag{91}$$

For a surface graph of $g(x,y)$, the surface integral is given by:

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y))\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dA \tag{92}$$

Normal Vectors:
For a parametric surface, the normal vector is given by:

$$\mathbf{n} = \frac{\mathbf{r_u} \times \mathbf{r_v}}{|\mathbf{r_u} \times \mathbf{r_v}|} \tag{93}$$

For a surface graph of $g(x,y)$, the normal vector is given by:

$$\mathbf{n} = \frac{-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}} \tag{94}$$

Flux:
If $\mathbf{F}$ is a vector field on a surface S with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over S is:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS \tag{95}$$

This integral is also called the flux of $\mathbf{F}$ across S. For a parametric surface, the flux is given by:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r_u} \times \mathbf{r_v})\, dA \tag{96}$$

For a surface graph of $g(x,y)$, the flux is given by:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \tag{97}$$

Advanced Vector Field Theorems

Curl:
If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then the curl of $\mathbf{F}$, denoted by curl $\mathbf{F}$ or also $\nabla \times \mathbf{F}$, is defined by:

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \tag{98}$$

If f is a function of three variables, then:

$$\mathrm{curl}(\nabla f) = \mathbf{0} \tag{99}$$

This implies that if $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \mathbf{0}$. The converse is only true if $\mathbf{F}$ is defined on all of $\mathbb{R}^n$. So if $\mathbf{F}$ is defined on all of $\mathbb{R}^n$ and if curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Divergence:
If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then the divergence of $\mathbf{F}$, denoted by div $\mathbf{F}$ or also $\nabla \cdot \mathbf{F}$, is defined by:

$$\mathrm{div}\ \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{100}$$

If $\mathbf{F}$ is a vector field on $\mathbb{R}^n$, then div curl $\mathbf{F} = 0$. If div $\mathbf{F} = 0$, then $\mathbf{F}$ is said to be incompressible. Note that curl $\mathbf{F}$ returns a vector field and div $\mathbf{F}$ returns a scalar field.

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Green's Theorem:
Let C be a positively oriented piecewise-smooth simple closed curve in the plane and D be the region bounded by C. Now:

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \quad (101)$$

This can also be useful for calculating areas. To calculate an area, take functions P and Q such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ and then apply Green's theorem.
In vector form, Green's theorem can also be written as:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dA \quad (102)$$

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \text{div } \mathbf{F}(x, y)\, dA \quad (103)$$

Stoke's Theorem:
Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field that contains S. Then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \quad (104)$$

The Divergence Theorem:
Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field on an open region that contains E. Then:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F}\, dV \quad (105)$$

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