Differential Equations

① $x\,dy + y\,dx = d(xy)$

② $\dfrac{x\,dy - y\,dx}{x^2} = d(y/x)$

③ $\dfrac{x\,dy - y\,dx}{y^2} = d(x/y)$

④ $\dfrac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

⑤ $\dfrac{x\,dy - y\,dx}{xy} = d\left(\log\left(\dfrac{y}{x}\right)\right)$.

⑥ $\dfrac{x\,dy - y\,dx}{x^2-y^2} = d\left[\dfrac{1}{2}\log\left(\dfrac{x+y}{x-y}\right)\right]$

⑦ $\dfrac{x\,dy + y\,dx}{x^2y^2} = d\left[-\dfrac{1}{xy}\right]$

⑧ $\dfrac{x\,dx + y\,dy}{x^2+y^2} = d\left[\dfrac{1}{2}\log\left(x^2+y^2\right)\right]$

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① **Homogeneous Equations:**

$\dfrac{dy}{dx} = \dfrac{f(x,y)}{\phi(x,y)}$ put $y = vx$. $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Ex:# $\dfrac{dy}{dx} = \dfrac{x^2 y^2}{xy}$

② **Equations Reducible to Homogeneous Form**

$$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$$

Case I: when $\dfrac{a}{A} \neq \dfrac{b}{B}$ then put $x = X+h$, $y = Y+K$ h, K constants

Case II when $\dfrac{a}{A} = \dfrac{b}{B} = \dfrac{1}{m}$ (say); put $ax+by = k(Ax+By)$

Put: $ax+by = t$

③ **Linear Differential Equations:** Leibnitz linear equation

$\dfrac{dy}{dx} + Py = Q$. where, P, Q are functions of x only. or constants.

$\left\{\begin{array}{l} T.F = e^{\int P dx} \\ \text{solution is } y \cdot I.F = \int Q \cdot I.F \cdot dx + C \end{array}\right\}$.

④ **Bernoulli's Equation**

$\dfrac{dy}{dx} + Py = Qy^n$

Divide by y^n. and put $\dfrac{1}{y^{n-1}} = z$ & solve as per LDE. (I.F. method).

(5) Exact Differential Equation.

Necessary and sufficient condition for the diff. equation $Mdx + Ndy = 0$ to be exact is

$$\left[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right]$$

Solution of $Mdx + Ndy$ is:—

$$\int_{y=\text{const.}} M dx + \int N dy = c$$

(terms of N not containing x)

Equations reducible to the exact equations

Rule 1. If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ alone. then. $I.F. = e^{\int f(x) dx}$

Rule 2 If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$ alone. then. $I.F = e^{\int f(y) dy}$.

Rule 3. If M is of the form $M = y f_1(xy)$ and $N = x f_2(xy)$

then

$$I.F = \left(\frac{1}{Mx - Ny}\right)$$

Rule 4

For the type of $x^m y^n (a y dx + b x dy) + x^{m'} y^{n'} (a' y dx + b' x dy) = 0$,

$I.F = x^h y^k$.

where,

$$\frac{m+h+1}{a} = \frac{n+k+1}{b} \quad \text{—①}$$

$$\frac{m'+h+1}{a'} = \frac{n'+k+1}{b'}$$

Rule 5 If the given equation $Mdx + Ndy = 0$ is a homogeneous equation and $Mx + Ny \neq 0$, then $I.F. = \dfrac{1}{Mx + Ny}$.

CLAIRAUT EQUATION:

$$\boxed{y = px + f(p)}$$

solution ⇒ $y = ax + f(a)$

Put $\boxed{p = a = \text{const}}$

HOMOGENEOUS LINEAR EQUATIONS

$$\boxed{a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = \phi(x).}$$

Put $x = e^z$ $\quad z = \log_e x.$ $\quad \frac{d}{dz} \equiv D.$

$$x\frac{dy}{dx} \equiv Dy. \quad ; \quad x^2\frac{d^2y}{dx^2} \equiv D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} \equiv D(D-1)(D-2)y.$$

Ex# $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$ $\quad x = e^z$

$\Rightarrow D(D-1)y - 2Dy - 4y = e^{4z}$

$\Rightarrow [D^2 - D - 2D - 4]\, y = e^{4z}$

$\Rightarrow (D^2 - 3D - 4)\, y = e^{4z}.$

A.E. $\Rightarrow D^2 - 3D - 4 = 0$

$D = -1, 4.$

C.F. $= c_1 e^{-z} + c_2 e^{4z}.$

$$P.I. = \frac{1}{D^2-3D-4} e^{4z} = \cancel{\frac{e^{4z}}{16-12-4}} =$$

$$= z \cdot \frac{e^{4z}}{2D-3} = \frac{z e^{4z}}{5}.$$

$$\boxed{\text{complete solution} = \frac{c_1}{x} + c_2 x^4 + \frac{1}{5} x^4 \log_e x}$$

Method of Variation of Parameters

It applies to this type of D.E. only;

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = X$$

C.F $= Ay_1 + By_2$ $\quad$ so that y_1 and y_2 satisfy $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$

P.I. $= uy_1 + vy_2.$

Known func of x

$$\boxed{u = \int \frac{-y_2 X}{y_1 y_2' - y_1' y_2}\, dx} \qquad \boxed{v = \int \frac{+y_1 X}{y_1 y_2' - y_1' y_2}\, dx.}$$

CASE I: When $X = e^{ax}$

(i) $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$. provided $f(a) \neq 0$.

otherwise

$\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} \cdot e^{ax}$. otherwise $\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}$, & so on.

case II: when $X = \sin(ax+b)$ or $\cos(ax+b)$.

$\frac{1}{f(D^2)} \cdot \sin(ax+b) = \frac{1}{f(-(a^2))} \sin(ax+b)$ provided $f(-a^2) \neq 0$.

otherwise,

$\frac{1}{f(D^2)} \sin(ax+b) = x \frac{1}{f'(-(a^2))} \sin(ax+b)$. & so on.

case III when $X = x^n$.

$P.I. = \frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$.
expand by the Binomial theorem.

case IV when. $X = e^{ax} \cdot \phi(x)$
ϕ being a function of x.

$P.I. = \frac{1}{f(D)} e^{ax} \cdot \phi(x) = \frac{1}{f(D+a)} \cdot \phi(x)$.

case V when X is any other function of 'x'

$P.I. = \frac{1}{f(D)} X$ if $f(D) = (D-m_1)(D-m_2) - \cdots (D-m_n)$.

General method

$\frac{1}{f(D)} X = \left[\frac{A_1}{(D-m_1)} + \frac{A_2}{(D-m_2)} + \cdots\cdots + \frac{A_n}{(D-m_n)}\right] X$

$= A_1 e^{m_1 x} \int X e^{-m_1 x} dx + A_2 . e^{m_2 x} \int X e^{-m_2 x} + \cdots\cdots + A_n e^{m_n x} \int e^{-m_n x} X dx$

special cases

case I: $\frac{1}{f(D)} x^n \sin ax$.

$\frac{1}{f(D)} x^n \sin ax =$ Imaginary part of $e^{iax} \cdot \frac{1}{f(D+ia)} x^n$.

$\frac{1}{f(D)} x^n \cos ax =$ Real part of $e^{iax} \cdot \frac{1}{f(D+ia)} x^n$.

Linear Differential Equations with constant coefficients

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + K_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + K_n y = 0.$$

$D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n = 0$ → Auxiliary Equation.

Let $m_1, m_2, \ldots, m_n$ be its roots.

CASE I: If all the roots be real and different;

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}}$$

CASE II: If two roots are equal; (i.e. $m_1 = m_2$).

$$\boxed{y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.}$$

If A.E. has three equal roots (i.e., $m_1 = m_2 = m_3$)

$$\boxed{y = (c_1 x^2 + c_2 x + c_3) e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}}$$

CASE III: If one pair of roots be imaginary then:

$m_1 = \alpha + i\beta$; $m_2 = \alpha - i\beta$.

$$\boxed{y = e^{\alpha x} [c_1 \cos\beta x + c_2 \sin\beta x] + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}}$$

CASE IV: If two pairs of imaginary roots be equal i.e. $m_1 = m_2 = \alpha + i\beta$ & $m_3 = m_4 = \alpha - i\beta$ then

$$\boxed{y = e^{\alpha x} [(c_1 x + c_2) \cos\beta x + (c_3 x + c_4) \sin\beta x] + \cdots + c_n e^{m_n x}}$$

INVERSE OPERATOR

① $f(D) \left\{ \frac{1}{f(D)} X \right\} = X$ ② $\frac{1}{D}(X) = \int X\,dx$. ③ $\frac{1}{D-a} X = e^{ax} \int X e^{-ax}\,dx$.

Rules for finding P.I.

consider the equation:-

$$\boxed{\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + K_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + K_n y = X}$$

Euler-Cauchy Equation

$$\boxed{x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0} \Rightarrow \text{Euler-Cauchy Equation}$$

we try $y = x^m$; $\frac{dy}{dx} = mx^{m-1}$. $\frac{d^2y}{dx^2} = m(m-1)\,x^{m-2}$.

$x^2 . x^{m-2}\,m(m-1) + ax\,m x^{m-1} + bx^m = 0$

$\Rightarrow x^m\left[m^2 + (a-1)m + b\right] = 0$

Auxiliary equation $\Rightarrow m^2 + (a-1)m + b = 0$

Case I: Distinct Real Roots (m_1, m_2).

$$\boxed{y = c_1 x^{m_1} + c_2 x^{m_2}}$$

Case II Double root $= \frac{1}{2}(1-a)$

$$y = \left(c_1 + c_2 \ln x\right) x^{\frac{(1-a)}{2}}$$

Ex # $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 0$

$a = -3$.

$x^{(1-(-3))/2} = x^2$.

$y = (c_1 + c_2 \ln x)\,x^2$

Case III complex conjugate roots

$m_1 = \alpha + i\beta \qquad m_2 = \alpha - i\beta$

$$\boxed{y = x^{\alpha}\left[A\cos(\beta \ln x) + B\sin(\beta \ln x)\right]}$$

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Ex: $x^2y'' + 7xy' + 13y = 0$